Name (last, first): ______________________

Student ID: ______________________

☐ **Write your name and PID on the top of EVERY PAGE.**

☐ **Write the solutions to each problem on separate pages. CLEARLY INDICATE on the top of each page the number of the corresponding problem. Different parts of the same problem can be written on the same page (for example, part (a) and part (b)).**

☐ **Remember this exam is graded by a human being. Write your solutions NEATLY AND COHERENTLY, or they risk not receiving full credit.**

☐ **You may assume that all transition probability functions are STATIONARY.**

☐ **You are allowed to use two 8.5 by 11 inch sheets of paper with handwritten notes (on both sides); no other notes (or books) are allowed.**

1. Let $(X_t)_{t\geq 0}$ be a birth and death process on states $\{0,1,2,3\}$ with state 0 absorbing, birth rates $\lambda_1 = 1$, $\lambda_2 = 3$ and the death rates $\mu_1 = 1$, $\mu_2 = 1$, $\mu_3 = 1$.

   (a) Draw the diagram of the jump chain of $(X_t)_{t\geq 0}$ and indicate the distribution of the sojourn times.

   (b) Suppose that X_0, the state of the process at time $t = 0$, is uniformly distributed on the set $\{1,2,3\}$. Compute the expectation of the time at which the process is absorbed at state 0.

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

2. Certain device consists of two components. The amount of time that the components work before breaking down has exponential distribution with rate 1. If any of the components fails, the repair time has exponential distribution with mean 2. The two components work independently and are repaired independently of each other.

   The number of components working at time t is given by the process $(X_t)_{t\geq 0}$ which is a continuous time Markov chain.

   (a) Determine the genetator Q of $(X_t)_{t\geq 0}$.

   (b) Determine the stationary distribution of $(X_t)_{t\geq 0}$.

   (c) In the long run, what fraction of time both components work simultaneously?

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

3. Let X and Y be random variables. Suppose that $X \sim \text{Exp}(2)$, and given $X = x$, Y is distributed on $[0, x]$ with linear density

$$f_{Y|X}(y|x) = \alpha_x y. \tag{1}$$

   (a) Determine α_x.

   (b) Compute $E(Y \mid X = x)$.

   (c) Compute $E(Y)$.

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

4. The economic history of a certain county is characterized by alternating periods of long economic growth and periods of long recession. Suppose that the lengths of all periods are independent and have uniform distribution on $[0, 1]$ (in years), both for growth and recession. At the beginning of our observation (time $t = 0$) a new recession starts.

   (a) Let X and Y be independent random variables having uniform distributions on [0,1]. Compute

   $$P(X + Y \le t) = \begin{cases} & t \le 0, \\ & 0 < t \le 1, \\ & 1 < t \le 2, \\ & t > 2. \end{cases}$$

   [Hint. Draw a unit square.]

   (b) What is the long-run probability that there will be no new recession starting within next year [Hint. Formulate using the excess life.]

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

5. Let $X_1, X_2, \ldots$ be *i.i.d.* random variables having exponential distribution with rate 1, i.e., $X_1 \sim \text{Exp}(1)$.

   (a) Let Y be an exponential random variable with rate λ. Compute

   $$M_Y(t) = E(e^{tY}) = \begin{cases} \quad , & t < \lambda, \\ \quad , & t \geq \lambda \end{cases} \tag{2}$$

   for $t \in (-\infty, \infty)$. (Recall that $M_Y(t)$ is called the moment generating function of Y).

   (b) Using the result from (a) show that for any $t < 1$, the process $(Z_n)_{n \geq 1}$ defined by

   $$Z_n = (1-t)^n e^{t \sum_{i=1}^n X_i}, \quad Z_0 = 1 \tag{3}$$

   is a nonegative martingale.

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

6. Let $(X_t)_{t\geq 0}$ be a Brownian motion with drift μ and variance parameter σ^2. It is given that $X_0 = 0$, $E(X_1) = 1$ and $\mathrm{Var}(X_1) = 1$.

   (a) Determine μ and σ^2.

   (b) Suppose that the price fluctuations of a share are modeled by the process $(Z_t)_{t\geq 0}$ given by

   $$Z_t = e^{X_t}. \tag{4}$$

   Determine the probability that the price of the share doubles before it drops by one half (i.e., probability that the price increases from 1 to 2 before in drops from 1 to 1/2).

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

7. The fluctuations of the cash assets of a certain company are modeled by a Brownian motion with variance parameter $\sigma^2 = 2$ reflected at 0 (taking only positive values). Suppose that initially (at time $t = 0$) the cash assets of the company are equal to 10.

   Determine the probability that **at time** $t = 50$ the cash assets do not exceed 20. [Express the answer in terms of the CDF of the standard normal distribution $\Phi(x)$.]

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

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