☐ **Write your name and PID on the top of EVERY PAGE.**

☐ **Write the solutions to each problem on separate pages. CLEARLY INDICATE on the top of each page the number of the corresponding problem.**

☐ **Remember this exam is graded by a human being. Write your solutions NEATLY AND COHERENTLY, or they risk not receiving full credit.**

☐ **You may assume that all transition probability functions are STATIONARY.**

1. (25 points) Let $(X_t)_{t\geq 0}$ be a continuous time Markov chain on the state space $\{0,1,2\}$ with transition probability functions

$$P(t) = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{lll} e^{-3t} & \frac{1}{2} - e^{-3t} + \frac{1}{2}e^{-4t} & \frac{1}{2} - \frac{1}{2}e^{-4t} \\ 0 & \frac{1}{2} + \frac{1}{2}e^{-4t} & \frac{1}{2} - \frac{1}{2}e^{-4t} \\ 0 & \frac{1}{2} - \frac{1}{2}e^{-4t} & \frac{1}{2} + \frac{1}{2}e^{-4t} \end{array} \right\| \end{array}. \tag{1}$$

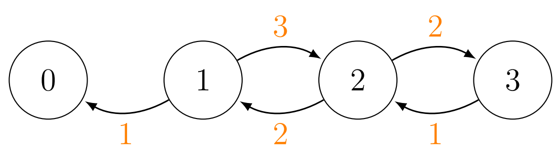
(a) (7 points) Compute the generator Q of $(X_t)_{t\geq 0}$. [Hint. Recall that $P'(0) = Q$.]

(b) (6 points) Give the jump-and-hold description of $(X_t)_{t\geq 0}$.

(c) (6 points) Draw the rate diagram of $(X_t)_{t\geq 0}$.

(d) (6 points) Assuming that X_0 is uniformly distributed on $\{0,1,2\}$, compute the probability that $X_1 = 2$.

2. (25 points) Let $(X_t)_{t\geq 0}$ be a birth and death process on $\{0,1,2,3\}$ described by the following rate diagram



Compute the mean time to absorption starting from state 1 (i.e., given $X_0 = 1$).

3. (25 points) Let $(X_t)_{t\geq 0}$ be a continuous time Markov chain on the state space $\{0,1,2\}$ with generator

$$Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} -2 & 2 & 0 \\ 3 & -6 & 3 \\ 0 & 1 & -1 \end{array} \right\| \end{array}. \tag{2}$$

(a) (10 points) Draw the diagram for the jump chain of $(X_t)_{t\geq 0}$ and explain why $(X_t)_{t\geq 0}$ is irreducible.

(b) (10 points) Compute the stationary distribution for $(X_t)_{t\geq 0}$.

(c) (5 points) What is the expected average fraction of time that $(X_t)_{t\geq 0}$ will spend in states 1 and 2 in the long run?

4. (25 points) Certain printing facility has two printers operating on a 24/7 basis and one repairman that takes care of the printers. The amount of time (in hours) that a printer works before breaking down has exponential distribution with mean 2. If a printer is broken, the repairman needs exponentially distributed amount of time with mean 1 (hour) to repair the broken printer. The repairman cannot repair two printers simultaneously. Each printer can produce 100 pages per minute.

   Let X_t denote the number of printers in operating state at time t.

   (a) (10 points) Assuming without proof that $(X_t)_{t\geq 0}$ is a Markov process, determine the generator of $(X_t)_{t\geq 0}$ (you can provide rigorous computations for only one entry of matrix Q.)

   [Hint. If $T \sim \text{Exp}(\gamma)$, then $P(T \leq h) = \gamma h + o(h)$ as $h \to 0$.]

   (b) (10 points) Compute the stationary distribution for $(X_t)_{t\geq 0}$.

   (c) (5 points) In the long run, how many pages does the facility produce on average per minute?