MATH180C: Introduction to Stochastic Processes II

Lecture A00: math.ucsd.edu/~ynemish/teaching/180cA
Lecture B00: math.ucsd.edu/~ynemish/teaching/180cB

Today: Asymptotic behavior of renewal processes

Next: PK 7.5, Durrett 3.1, 3.3

Week 7:

- homework 6 (due Monday, May 16, week 8)

Key renewal theorem

Suppose $H(t)$ is an unknown function that satisfies

$$H(t) = h(t) + H*F(t) \quad (*)$$

renewal equation

E.g.: $M(t) = F(t) + M*F(t)$,

$$m(t) = f(t) + m*F(t) = f(t) + m*f(t)$$

Remark about notation

- Convolution with c.d.f.: $g*F(t) = \int_{-\infty}^{+\infty} g(t-x)\, dF(x)$
- Convolution with p.d.f.: $g*f(t) = \int_{-\infty}^{+\infty} g(t-x) f(x)\, dx$

__Def__. Function $h: [0,+\infty) \to \mathbb{R}$ is called locally bounded if $\max\limits_{0\le x\le t} f(x) < \infty \ \forall t$

__Def.__ Function h is absolutely integrable if

$$\int_0^\infty |h(x)|\, dx < \infty$$

Key renewal theorem

Thm (Key renewal theorem) Let h be locally bounded.

(a) If H satisfies $H = h + h * M$, then H is locally bounded

and $$H = h + H * F \quad (*)$$

(b) Conversely, if H is a locally bounded solution to $(*)$,

then $$H = h + h * M \quad (**)$$ [convolution in the Riemann-Stieltjes sense]

(c) If h is absolutely integrable, then

$$\lim_{t\to\infty} H(t) = \frac{\int_0^\infty h(x)\,dx}{\mu}$$

No proof.

Remark. Key renewal theorem says that if h is locally bounded, then there exists a unique locally bounded solution to $(*)$ given by $(**)$

Examples

- Renewal function: $M(t)$ satisfies

and $$M = F + M * F \;=\; F + F * M$$
$$H = h + H * F \qquad h + h * M$$

$F(t)$ is nondecreasing, so (c) does not apply to the renewal equation for $M(t)$

- Renewal density: $m(t)$ satisfies

and $$m = f + m * F = f + m * f = f + f * m$$
$$= f + f * M \text{ (in the Riemann - Stieltjes sense)}$$

f is absolutely integrable, $\int_0^\infty f(x)\,dx = 1$, so

$$\lim_{t\to\infty} m(t) = \frac{\int_0^\infty f(x)\,dx}{\mu} = \frac{1}{\mu}$$

Important remark

Let $W = (W_1, W_2, \dots)$ be arrival times of a renewal process, and denote $W' = (W_1', W_2', \dots)$ with

$$W_i' = W_{i+1} - W_1 = X_2 + X_3 + \dots + X_{i+1},$$

shifted arrival times.

Then:

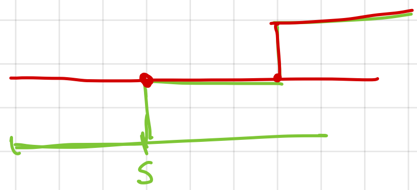
- W' is independent of $W_1 = X_1$, and
- W' has the same distribution as W

Example

Example. Compute $\lim_{t\to\infty} E(\gamma_t)$. Take $H(t)=E(\gamma_t)$

If $X_1>t$, then $\gamma_t = X_1 - t$; if $X_1<t$ condition on $X_1=s$

$$E(\gamma_t) = E\left((X_1-t)\mathbb{1}_{X_1>t}\right) + E\left(\gamma_t \mathbb{1}_{X_1\le t}\right)$$



$$E(\gamma_t \mathbb{1}_{X_1\le t}) = \int_0^\infty P\left((W_{N(t)+1}-t)\mathbb{1}_{X_1\le t} > w\right)dw$$

$$= \int_0^\infty \sum_{k=1}^\infty P\left((W_k - t)\mathbb{1}_{X_1\le t} > w,\ N(t)=k-1\right)dw$$

$$= \int_0^\infty \sum_{k=2}^\infty P\left(\left(X_1+\sum_{j=2}^k X_j - t\right)\mathbb{1}_{X_1\le t}>w,\ N(t)=k-1\right)dw$$

$$= \int_0^\infty \left[\sum_{k=2}^\infty \int_0^t P\left(\overbrace{\sum_{j=2}^k X_j}^{W'_{k-1}} - (t-s) > w,\ N(t)=k-1\right)dF(s)\right]dw$$

W'_{k-1} , $k-1=:\ell$ $\quad\iff N'(t-s)=k-2$

$$= \int_0^t \left[\int_0^\infty \underbrace{\sum_{\ell=1}^\infty P\left(W'_\ell-(t-s)>w,\ N'(t-s)=\ell-1\right)dw}_{P(\gamma'_{t-s}>w)}\right]dF(s) = \int_0^t E(\gamma_{t-s})\,dF(s) = H*F$$

Example (cont)

Assume that $E(X_1)=\mu$, $Var(X_1)=\sigma^2$

$$E\left((X_1-t)\mathbb{1}_{X_1>t}\right) = \int_t^\infty (x-t)\,dF(x) = \int_t^\infty (t-x)\,d(1-F(x))$$

$$= \underbrace{(t-x)(1-F(x))\Big|_t^\infty}_{0} + \int_t^\infty (1-F(x))\,dx$$

Since we assume that $Var(X_1)=\sigma^2$,

and Ex: $x(1-F(x)) \to 0$, as $x\to\infty$

Finally, we have that

$$H(t) = \int_t^\infty (1-F(x))\,dx + H * F$$

therefore $H(t) = h(t) + h * M(t)$

with $h(t) = \int_t^\infty (1-F(x))\,dx$