MATH180C: Introduction to Stochastic Processes II

Lecture A00: math.ucsd.edu/~ynemish/teaching/180cA
Lecture B00: math.ucsd.edu/~ynemish/teaching/180cB

Today: Asymptotic behavior of renewal processes

Next: PK 7.5, Durrett 3.1, 3.3

Week 7:

- homework 6 (due Monday, May 16, week 8)

Key renewal theorem

Suppose $H(t)$ is an unknown function that satisfies

$$H(t) = h(t) + H * F(t) \quad (*)$$

renewal equation

E.g.: $M(t) = F(t) + M * F(t)$,

$m(t) = f(t) + m * F(t) = f(t) + m * f(t)$

Remark about notation

- Convolution with c.d.f.: $g * F(t) = \int_{-\infty}^{+\infty} g(t-x)\, dF(x)$
- Convolution with p.d.f.: $g * f(t) = \int_{-\infty}^{+\infty} g(t-x) f(x)\, dx$

<u>Def</u>. Function h is called locally bounded if

<u>Def</u>. Function h is absolutely integrable if

Key renewal theorem

Thm (Key renewal theorem) Let h be locally bounded.

(a) If H satisfies , then H is locally bounded

and

(b) Conversely, if H is a locally bounded solution to $(*)$,

then [convolution in the Riemann-Stieltjes sense]

(c) If h is absolutely integrable , then

No proof.

Remark. Key renewal theorem says that if h is locally bounded, then there exists a unique locally bounded solution to $(*)$ given by $(**)$

Examples

- Renewal function: $M(t)$ satisfies

  and

  $F(t)$ is nondecreasing, so (c) does not apply to the renewal equation for $M(t)$

- Renewal density: $m(t)$ satisfies

  and

  (in the Riemann-Stieltjes sense)

  f is absolutely integrable, , so

Important remark

Let $W = (W_1, W_2, \dots)$ be arrival times of a renewal process, and denote $W' = (W_1', W_2', \dots)$ with

$$W_i' = W_{i+1} - W_1 = X_2 + X_3 + \dots + X_{i+1},$$

shifted arrival times.

Then:

- W'
- W'

Example

Example. Compute $\lim_{t\to\infty} E(\gamma_t)$. Take $H(t) = E(\gamma_t)$

If $X_1 > t$, then ; if $X_1 < t$ condition on $X_1 = s$

$E(\gamma_t) =$

$E(\gamma_t \mathbb{1}_{X_1 \le t}) =$

$=$

$=$

$=$

$=$

Example (cont)

Assume that $E(X_1)=\mu$, $Var(X_1)=\sigma^2$

$$E((X_1-t)\mathbb{1}_{X_1>t}) =$$

$$=$$

Since we assume that $E(X_1)=\sigma^2$,

and

Finally, we have that

$$H(t) =$$

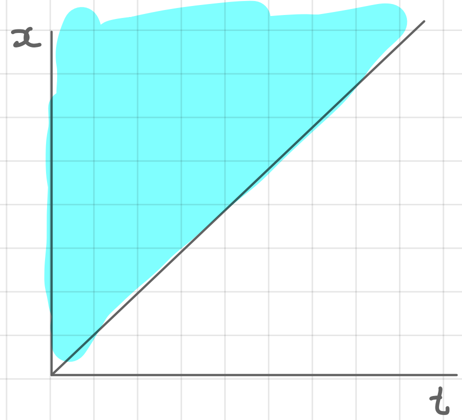
therefore $H(t) = h(t) + h * M(t)$

with $h(t) =$

Example (cont)

In particular,

$$\int_0^\infty \int_t^\infty (1-F(x))\,dx\,dt =$$



$\Rightarrow$ by part (c) of the key renewal theorem

$$\lim_{t\to\infty} E(\gamma_t) =$$

Similarly $\lim_{t\to\infty} E(\delta_t) =$, $\lim_{t\to\infty} E(\beta_t) =$

Example

What is the expected time to the next earthquake in the long run?

For $X_1 \sim \text{Unif}[0,1]$

therefore, $\lim_{t\to\infty} E(\gamma_t) =$

And the long run expected time between two consecutive earthquakes is