MATH180C: Introduction to Stochastic Processes II

Lecture A00: math.ucsd.edu/~ynemish/teaching/180cA
Lecture B00: math.ucsd.edu/~ynemish/teaching/180cB

Today: Martingales

Next: PK 8.1

Week 8:

- homework 6 (due Monday, May 16, week 8)
- OH Tuesday 3-4:30 PM, APM 7218

Midterm 2: Wednesday, May 18

Martingales

Definition. A stochastic process $(X_n, n \geq 0)$ is a martingale if for $n = 0, 1, \dots$

(a) $E(|X_n|) < \infty \quad \forall n$

(b) $E(X_{n+1} \mid X_0, X_1, \dots, X_n) = X_n$

After taking the expectation of both sides of (b) we get that

$$E(X_{n+1}) = E(X_n)$$

$(X_n)_{n \geq 0}$ is a martingale $\Rightarrow E(X_n) = E(X_0) \quad \forall n$

- submartingale: $E(X_{n+1} \mid X_0, \dots, X_n) \geq X_n$ (increases)
- supermartingale: $E(X_{n+1} \mid X_0, \dots, X_n) \leq X_n$ (decreases)

Examples of martingales

(i) Let $X_1, X_2, \dots$ be independent RV's with $E(|X_k|) < \infty$ and $E(X_k) = 0$. Define $S_n = X_1 + \dots + X_n$, $S_0 = 0$.

Then
$$E(S_{n+1} \mid S_0, \dots, S_n) = E(S_n + X_{n+1} \mid S_0, \dots, S_n)$$
$$= E(S_n \mid S_0, \dots, S_n) + E(X_{n+1} \mid S_0, \dots, S_n)$$
$$= S_n + E(X_{n+1}) = S_n \qquad \forall n$$

$\Rightarrow (S_n)_{n \geq 0}$ is a martingale with $E(S_n) = E(S_0) = 0$

(ii) Let $X_1, X_2, \dots$ be independent RV with $X_k \geq 0$, $E(|X_k|) < \infty$ and $E(X_k) = 1$. Define $M_n = X_1 X_2 \cdots X_n$, $M_0 = 1$.

Then
$$E(M_{n+1} \mid M_0, \dots, M_n) = E(M_n \cdot X_{n+1} \mid M_0, \dots, M_n)$$
$$= M_n E(X_{n+1} \mid M_0, \dots, M_n) = M_n \cdot E(X_{n+1}) = M_n$$

$\Rightarrow (M_n)_{n \geq 0}$ is a martingale with $E(M_n) = E(M_0) = 1$

Example

Stock prices in a perfect market

Let X_n be the closing price at the end of day n of a certain publicly traded security such as a share or stock. Many scholars believe that in a perfect market these price sequences should be martingales. (see PK page 73 for more details).

History and gambling

Let $(X_n)_{n\geq 0}$ be a stochastic process describing your total winnings in n games with unit stake.

Think of $X_n - X_{n-1}$ as your net winnings per unit stake in game n, $n \geq 1$, in a series of games, played at times $n = 1, 2, \dots$.

In the martingale case

$$E(X_n - X_{n-1} \mid X_0, \dots, X_{n-1}) = E(X_n \mid X_0, \dots, X_{n-1}) - E(X_{n-1} \mid X_0, \dots, X_{n-1})$$

$$= E(X_n \mid X_0, \dots, X_{n-1}) - X_{n-1} = 0 \quad (\text{fair game})$$

Some early works of martingales was motivated by gambling. Note that there exists a betting strategy called the "martingale system" ← doubling bets after losses

Some basic properties

Let $(X_n)_{n\geq 0}$ be a martingale.

- $E(X_m \mid X_0, \ldots, X_n) = X_n$ $\quad m > n$

Proof $\quad X_n = E(X_{n+1} \mid X_0, \ldots, X_n)$

$\quad X_{n+1} = E(X_{n+2} \mid X_0, \ldots, X_{n+1})$

$$X_n = E(X_{n+1} \mid X_0, \ldots, X_n) = E(E(X_{n+2} \mid X_0, \ldots, X_{n+1}) \mid X_0, \ldots, X_n)$$

$$= E(X_{n+2} \mid X_0, \ldots, X_n)$$

Exercise: $E(E(X \mid Y, Z) \mid Z) = E(X \mid Z)$ (show for discrete r.v.)

- Markov inequality: If $X_n \geq 0$ $\forall n$, then for any $\lambda > 0$

$$P(X_n \geq \lambda) \leq \frac{E(X_n)}{\lambda} = \frac{E(X_0)}{\lambda}$$

$\Rightarrow$ For all n $\quad P(X_n \geq \lambda) \leq \dfrac{E(X_0)}{\lambda}$

Maximal inequality for nonnegative martingales

Thm. Let $(X_n)_{n\geq 0}$ be a martingale with nonnegative values.

For any $\lambda>0$ and $m\in\mathbb{N}$

$$P\left(\max_{0\leq n\leq m} X_n \geq \lambda\right) \leq \frac{E(X_0)}{\lambda} \qquad (1)$$

and

$$P\left(\max_{n\geq 0} X_n \geq \lambda\right) \leq \frac{E(X_0)}{\lambda} \qquad (2)$$

Proof. We prove (1), (2) follows by taking the limit $m\to\infty$.

Take the vector $(X_0, X_1, \dots, X_m)$ and partition the sample space wrt the index of the first r.v. rising above λ

$$1 = \mathbb{1}_{X_0\geq\lambda} + \mathbb{1}_{X_0<\lambda, X_1\geq\lambda} + \cdots + \mathbb{1}_{X_0<\lambda,\dots,X_{m-1}<\lambda, X_m\geq\lambda} + \mathbb{1}_{X_0<\lambda,\dots,X_m<\lambda}$$

Compute $E(X_m) = E(X_m\cdot 1)$ using the above partition

Proof of the maximal inequality

$$E(X_m) = \sum_{n=0}^{m} E(X_m \mathbb{1}_{X_0<\lambda,\dots,X_{n-1}<\lambda,X_n\geq\lambda}) + \underbrace{E(X_m \mathbb{1}_{X_0<\lambda,\dots,X_m<\lambda})}^{\geq 0}$$

$$\geq \sum_{n=0}^{m} E(X_m \mathbb{1}_{X_0<\lambda,\dots,X_{n-1}<\lambda,X_n\geq\lambda})$$

Compute $E(X_m \mathbb{1}_{X_0<\lambda,\dots,X_{n-1}<\lambda,X_n\geq\lambda})$ by conditioning on $X_0, X_1, \dots, X_{n-1}, X_n$:

$$E(X_m \mathbb{1}_{X_0<\lambda,\dots,X_{n-1}<\lambda,X_n\geq\lambda})$$

$=$

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Sum for all n

$E(X_m) \geq$