MATH180C: Introduction to Stochastic Processes II

Lecture A00: math.ucsd.edu/~ynemish/teaching/180cA
Lecture B00: math.ucsd.edu/~ynemish/teaching/180cB

Today: Brownian motion

Next:

Week 10:

CAPES

- homework 8 (due Friday, June 3)
- HW7 regrades are active on Gradescope until June 4, 11 PM
- homework 9 and solutions are available on the course website

OH M: 6-7 PM, T: 5-7 PM APM 5829

Reflected BM

Def. Let $(B_t)_{t\geq 0}$ be a standard BM. The stochastic process

$$R_t = |B_t| = \begin{cases} B_t & \text{, if } B_t \geq 0 \\ -B_t & \text{, if } B_t < 0 \end{cases}$$

is called reflected BM.

Think of a movement in the vicinity of a boundary.

Moments: $E(R_t) = \int_{-\infty}^{+\infty} |x| \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx = 2 \cdot \int_0^{\infty} x \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx = \sqrt{\frac{2t}{\pi}}$

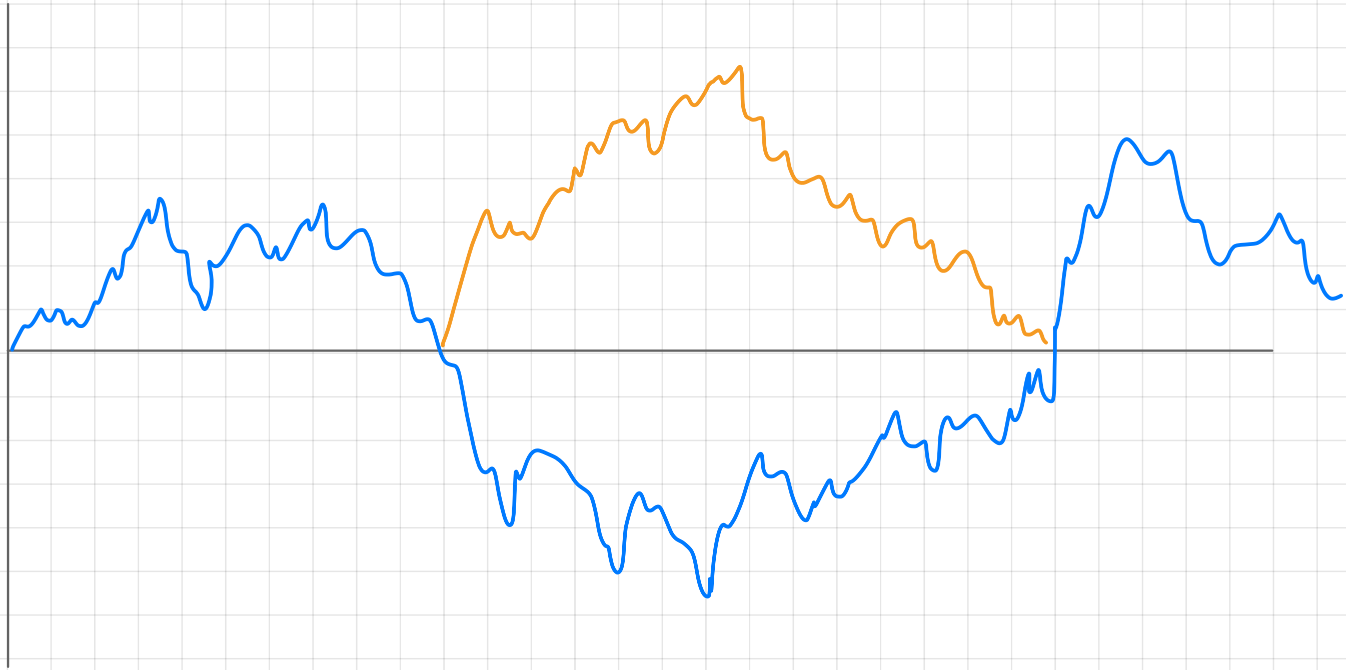
$$Var(R_t) = E(B_t^2) - \left(E(|B_t|)\right)^2 = t - \frac{2t}{\pi} = t\left(1 - \frac{2}{\pi}\right)$$

Transition density: $P(R_t \leq y \mid R_0 = x) = P(-y \leq B_t \leq y \mid B_0 = x)$

$$= \int_{-y}^{y} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-s)^2}{2t}} ds \Rightarrow p_t(x,y) = \frac{1}{\sqrt{2\pi t}} \left(e^{-\frac{(x-y)^2}{2t}} + e^{-\frac{(x+y)^2}{2t}} \right)$$

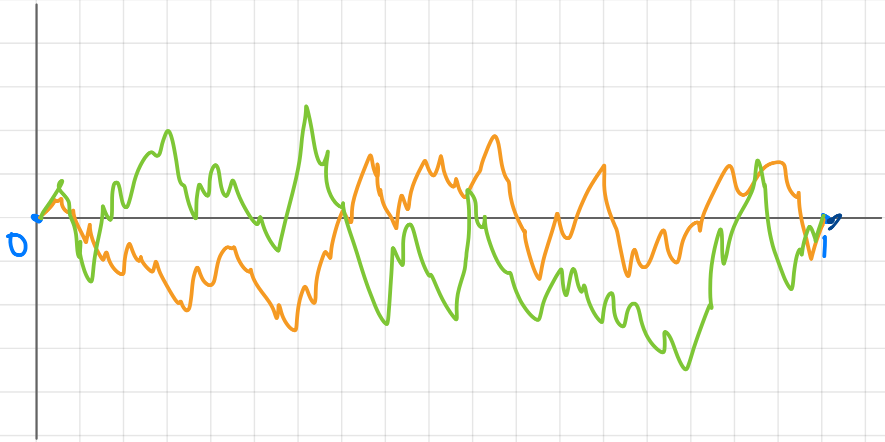
Thm (Lévy) Let $M_t = \max_{0\leq u\leq t} B_u$. Then $(M_t - B_t)_{t\geq 0}$ is a reflected BM.

Reflected BM



Brownian bridge

Brownian bridge is constructed from a BM by conditioning on the event $\{B(0)=0, B(1)=0\}$.



<u>Thm 1.</u> Brownian bridge is a continuous Gaussian process on $[0,1]$ with mean 0 and covariance function

$$\Gamma(s,t) = \min\{s,t\} - st$$

Brownian motion with drift

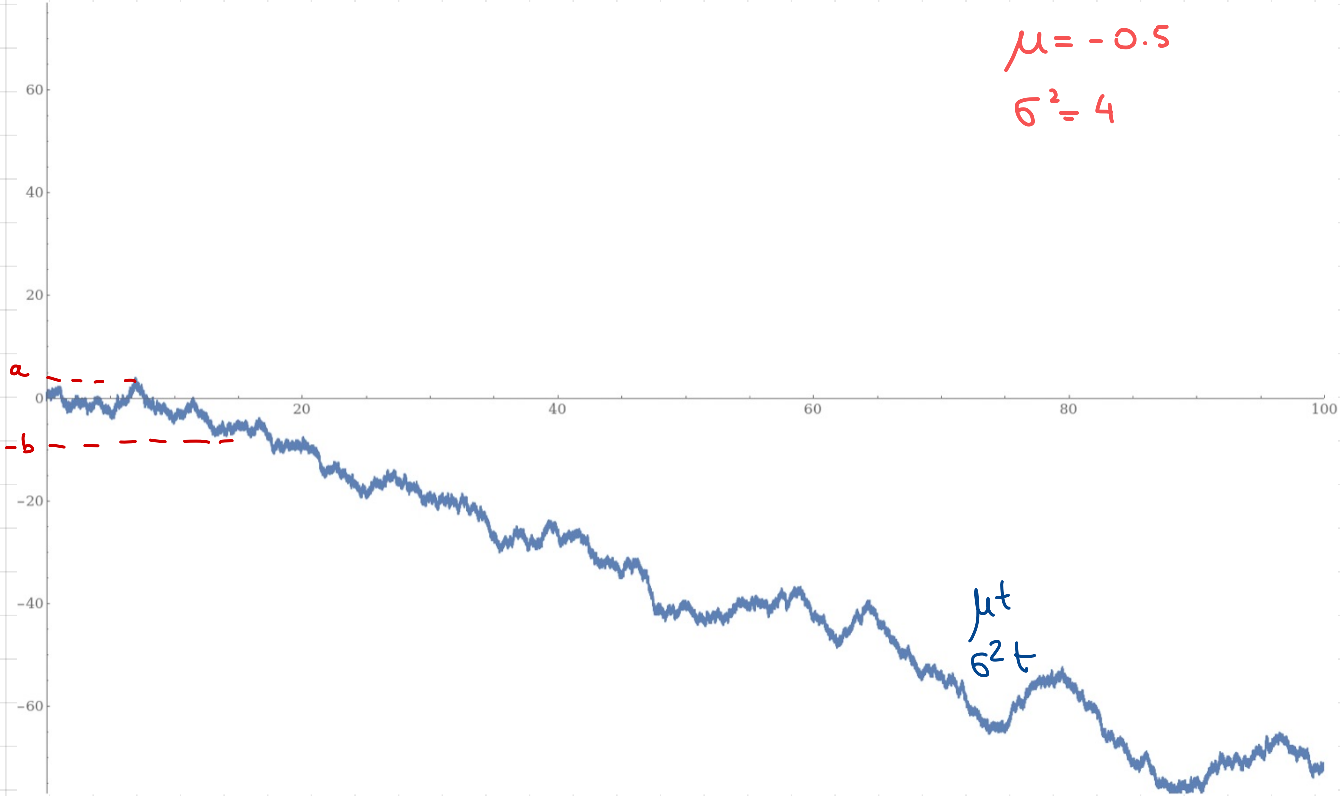
<u>Def</u> Let $(B_t)_{t\geq 0}$ be a standard BM. Then for $\mu \in \mathbb{R}$ and $\sigma > 0$ the process $(X_t)_{t\geq 0}$ with $X_t = \mu t + \sigma B_t$, $t \geq 0$ is called the Brownian motion with drift μ and variance paremeter σ^2.

<u>Remark</u> BM with drift μ and variance paremeter σ^2 is a stochastic process $(X_t)_{t\geq 0}$ satisfying

1) $X_0 = 0$, $(X_t)_{t\geq 0}$ has continuous sample paths
2) $(X_t)_{t\geq 0}$ has independent increments
3) For $t > s$ $\quad X_t - X_s \sim \mathcal{N}(\mu(t-s), \sigma^2(t-s))$

In particular, $X_t \sim \mathcal{N}(\mu t, \sigma^2 t) \Rightarrow X_t$ is not centered, not symmetric w.r.t. the origin

Brownian motion with drift



Gambler's ruin problem for BM with drift

Let $(X_t)_{t\geq 0}$ be a BM with drift $\mu \in \mathbb{R}$ and variance parameter $\sigma^2 > 0$. Fix $a < x < b$ and denote

$$T = T_{ab} = \min\{t \geq 0 : X_t = a \text{ or } X_t = b\}, \text{ and}$$

$$u(x) = P(X_T = b \mid X_0 = x).$$

Theorem.

(i) $$u(x) = \frac{\exp\left(-\frac{2\mu}{\sigma^2} x\right) - \exp\left(-\frac{2\mu}{\sigma^2} a\right)}{\exp\left(-\frac{2\mu}{\sigma^2} b\right) - \exp\left(-\frac{2\mu}{\sigma^2} a\right)}$$

(ii) $$E(T_{ab} \mid X_0 = x) = \frac{1}{\mu}\Big(u(x)(b-a) - (x-a)\Big)$$

No proof

$$\left(\underset{\uparrow \text{ SBM}}{u(x)} = \frac{b-x}{b-a} \right)$$

Example

Fluctuations of the price of a certain share is modeled by the BM with drift $\mu = 1/10$ and variance $\sigma^2 = 4$. You buy a share at 100 \$ and plan to sell it if its price increases to 110 \$ or drops to 95 \$.

(a) What is the probability that you will sell at profit?

(b) What is the expected time until you sell the share?

Denote by $(X_t)_{t \geq 0}$ a BM with drift $\frac{1}{10}$ and variance 4, $x = 100$, $b = 110$, $a = 95$. Then $2\mu/\sigma^2 = \frac{2 \cdot 0.1}{4} = \frac{1}{20}$ and

(a) $$P(X_T = 110 \mid X_0 = 100) = \frac{e^{-\frac{1}{20} \cdot 100} - e^{-\frac{1}{20} \cdot 95}}{e^{-\frac{1}{20} \cdot 110} - e^{-\frac{1}{20} \cdot 95}} \approx 0.419$$

(b) $$E(T \mid X_0 = 100) \approx \frac{1}{0.1}\Big(0.419\,(110 - 95) - (100 - 95)\Big) \approx 12.88$$

Maximum of a BM with negative drift

Thm Let $(X_t)_{t\geq 0}$ be a BM with drift $\mu<0$, variance σ^2 and $X_0=0$. Denote $M=\max_{t\geq 0} X_t$. Then

$$M \sim \mathrm{Exp}(-2\mu/\sigma^2)$$

Proof. $X_0=0$, therefore $M\geq 0$. For any $b>0$

$$P(M>b) = P\Big(\bigcup_{n\geq 1}\{X \text{ hits } b \text{ before } -n\}\Big)$$

$$= \lim_{n\to\infty} P(X \text{ hits } b \text{ before } -n)$$

$$= \lim_{n\to\infty} \frac{1-e^{2n\mu/\sigma^2}}{e^{-2b\mu/\sigma^2}-e^{2n\mu/\sigma^2}} = \frac{1}{e^{-2b\mu/\sigma^2}} = e^{-2b\mu/\sigma^2}$$

$$P(M>b) = e^{-(-2\mu/\sigma^2)b} \implies M \sim \mathrm{Exp}(-2\mu/\sigma^2)$$

$\blacksquare$

Geometric BM

Def. Stochastic process $(Z_t)_{t\geq 0}$ is called a geometric Brownian motion with drift parameter α and variance σ^2 if $X_t = \log Z_t$ is a BM with drift $\mu = \alpha - \frac{1}{2}\sigma^2$ and variance σ^2.

In other words, $Z_t = z \cdot e^{(\alpha - \frac{1}{2}\sigma^2)t + \sigma B_t}$, where $(B_t)_{t\geq 0}$ is a standard BM and $z > 0$ is the starting point $Z_0 = z$.

If $0 \leq t_1 < t_2 < \dots < t_n$, then $\frac{Z_{t_i}}{Z_{t_{i-1}}} = e^{(\alpha - \frac{1}{2}\sigma^2)(t_i - t_{i-1}) + \sigma(B_{t_i} - B_{t_{i-1}})}$

Since B has independent increments

$\frac{Z_{t_1}}{Z_{t_0}}, \frac{Z_{t_2}}{Z_{t_1}}, \dots, \frac{Z_{t_n}}{Z_{t_{n-1}}}$ are independent and

$$\frac{Z_{t_n}}{Z_{t_0}} = \frac{Z_{t_1}}{Z_{t_0}} \cdot \frac{Z_{t_2}}{Z_{t_1}} \cdot \dots \cdot \frac{Z_{t_n}}{Z_{t_{n-1}}} \leftarrow \text{"relative change of price = product of indepentent relative changes"}$$

Expectation of Geometric BM

Let $(Z_t)_{t\geq 0}$ be geometric BM with paremeters α and σ.

Then

$$E(Z_t \mid Z_0 = z) = E\left(z e^{(\alpha - \frac{1}{2}\sigma^2)t + \sigma B_t}\right) = z e^{(\alpha - \frac{1}{2}\sigma^2)t} E\left(e^{\sigma B_t}\right)$$

$$E(e^{\sigma B_t}) = e^{\frac{t\sigma^2}{2}}$$

$$\Rightarrow E(Z_t \mid Z_0 = z) = z e^{(\alpha - \frac{1}{2}\sigma^2)t} e^{t\frac{\sigma^2}{2}} = z e^{\alpha t}$$

Remark

It can be shown that for $0 < \alpha < \frac{1}{2}\sigma^2$ $Z_t \to 0$ as $t \to \infty$

At the same time, for $\alpha > 0$ $E(Z_t) \to \infty$.

Variance of geometric BM

$$E(Z_t^2 \mid Z_0 = z) = E\left(z^2 e^{2X_t}\right) = E\left(z^2 e^{(2\alpha-\sigma^2)t} e^{2\sigma B_t}\right)$$

$$= z^2 e^{(2\alpha-\sigma^2)t} e^{2\sigma^2 t} = z^2 e^{2\alpha t + \sigma^2 t}$$

$$\operatorname{Var}(Z_t \mid Z_0 = z) = z^2 e^{2\alpha t + \sigma^2 t} - z^2 e^{2\alpha t} = z^2 e^{2\alpha t}\left(e^{\sigma^2 t} - 1\right)$$

Theorem.

Let $(Z_t)_{t\geq 0}$ be geometric BM with paremeters α and σ^2.

Then

(i) $E(Z_t \mid Z_0 = z) = z e^{\alpha t}$

(ii) $\operatorname{Var}(Z_t \mid Z_0 = z) = z^2 e^{2\alpha t}\left(e^{\sigma^2 t} - 1\right)$

Gambler's ruin for geometric BM

Let $(Z_t)_{t\geq 0}$ be geometric BM with paremeters α and σ^2.

Let $A<1<B$, and denote $T=\min\{t : \frac{Z_t}{Z_0}=A \text{ or } \frac{Z_t}{Z_0}=B\}$.

Theorem

$$P\left(\frac{Z_T}{Z_0}=B\right)=\frac{1-A^{1-\frac{2\alpha}{\sigma^2}}}{B^{1-\frac{2\alpha}{\sigma^2}}-A^{1-\frac{2\alpha}{\sigma^2}}}$$

Example Fluctuations of the price are modeled by a geometric BM with drift $\alpha=0.1$ and variance $\sigma^2=4$. You buy a share at 100\$ and plan to sell it if its price increases to 110\$ or drops to 95\$.

Take $A=0.95$, $B=1.1$, $2\alpha/\sigma^2=\frac{1}{20}$, $1-2\alpha/\sigma^2=\frac{19}{20}=0.95$

$$P(X_T=110 \mid X_0=100)=\frac{1-0.95^{0.95}}{1.1^{0.95}-0.95^{0.95}}\approx 0.334$$