MATH180C: Introduction to Stochastic Processes II

Lecture A00: math-old.ucsd.edu/~ynemish/teaching/180cA
Lecture B00: math-old.ucsd.edu/~ynemish/teaching/180cB

Today: Birth processes. Yule process

Next: PK 6.2-6.3

Week 1:

- visit course web site
- homework 0 (due Friday April 1)
- join Piazza

Continuous Time Markov Chains

Def (Discrete-time Markov chain)

Let $(X_n)_{n\geq 0}$ be a discrete time stochastic process taking values in $\mathbb{Z}_+ = \{0,1,2,\dots\}$ (for convenience). $(X_n)_{n\geq 0}$ is called Markov chain if for any $n\in\mathbb{N}$ and $i_0, i_1, \dots, i_{n-1}, i, j \in \mathbb{Z}_+$

$$P(X_{n+1}=j \mid X_0=i_0, X_1=i_1, \dots, X_{n-1}=i_{n-1}, X_n=i) = P(X_{n+1}=j \mid X_n=i)$$

Def (Continuous-time Markov chain)

Let $(X_t)_{t\geq 0} = (X_t : 0\leq t<\infty)$ be a continuous time process taking values in $\mathbb{Z}_+$. $(X_t)_{t\geq 0}$ is called Markov chain if for any $n\in\mathbb{N}$, $0\leq t_0<t_1<\cdots<t_{n-1}<s$, $t>0$, $i_0, i_1, \dots, i_{n-1}, i, j\in\mathbb{Z}_+$

$$P(X_{s+t}=j \mid X_{t_0}=i_0, X_{t_1}=i_1, \dots, X_{t_{n-1}}=i_{n-1}, X_s=i) \overset{(*)}{=} P(X_{s+t}=j \mid X_s=i)$$

Transition probability function

One way of describing a continuous time MC is by using the transition probability functions.

Def. Let $(X_t)_{t\geq 0}$ be a MC. We call

$$P(X_{s+t}=j \mid X_s=i), \quad i,j\in\{0,1,\dots\}, \quad s\geq 0, \; t>0$$

the transition probability function for $(X_t)_{t\geq 0}$.

If $P(X_{s+t}=j \mid X_s=i)$ does not depend on s, we say that $(X_t)_{t\geq 0}$ has stationary transition probabilities and we define $P_{ij}(t) := P(X_{s+t}=j \mid X_s=i) \; \big(= P(X_t=j \mid X_0=i)\big)$

[compare with n-step transition probabilities]

Characterization of the Poisson process

Experiment: count events occurring along $[0,+\infty)$ $\left\{ \begin{array}{l} \text{time} \\ \text{or 1-D space} \end{array} \right.$

Denote by $N((a,b])$ the number of events that occur on $(a,b]$.

Assumptions:

1. Number of events happening in disjoint intervals are independent.
2. For any $t \geq 0$ and $h > 0$, the distribution of $N((t,t+h])$ does not depend on t (only on h, the length of the interval)
3. There exists $\lambda > 0$ s.t. $P(N((t,t+h]) \geq 1) = \lambda h + o(h)$ as $h \to 0$
   (rare events)
4. Simultaneous events are not possible: $P(N((t,t+h]) \geq 2) = o(h), h \to 0$

Transition probabilities of the Poisson process

Let $(X_t)_{t\geq 0}$ be the Poisson process.

Define the transition probability functions

$$P_{ij}(h) := P(X_{t+h} = j \mid X_t = i), \quad i,j \in \{0,1,2,\dots\}, \quad t \geq 0, \; h > 0$$

What are the infinitesimal (small h) transition probability functions for $(X_t)_{t\geq 0}$? As $h \to 0$,

$$P_{ii}(h) = P(X_{t+h} = i \mid X_t = i)$$

$$=$$

$$P_{i,i+1}(h) = P(X_{t+h} = i+1 \mid X_t = i) =$$

$$\sum_{j \notin \{i,i+1\}} P_{ij}(h) =$$

Poisson process and transition probabilities

To sum up: $(X_t)_{t\geq 0}$ is a MC with (infinitesimal) transition probabilities satisfying

$$P_{ii}(h) =$$

$$P_{i,i+1}(h) =$$

$$\sum_{j\notin\{i,i+1\}} P_{i,j}(h) =$$

What if we allow $P_{ij}(h)$ depend on i?

↳ birth and death processes

Pure birth processes

Def Let $(\lambda_k)_{k\geq 0}$ be a sequence of positive numbers.
We define a pure birth process as a Markov process $(X_t)_{t\geq 0}$ whose stationary transition probabilities satisfy

1. $P_{k,k+1}(h) =$
2. $P_{k,k}(h) =$
3. $P_{k,j}(h) =$
4. $X_0 = 0$

Related model. Yule process : $\lambda_k = \beta k$ for some $\beta > 0$.

Describes the growth of a population

- birth rate is proportional to the size of the population

Birth processes and related differential equations

Now define $P_n(t) = P(X_t = n)$. For small $h > 0$

$P_n(t+h) = P(X_{t+h} = n) =$

$=$

$=$

$=$

$=$

$$P_n(t+h) - P_n(t) = -\lambda_n h P_n(t) + \lambda_{n-1} h P_{n-1}(t) + o(h)$$

$P_n'(t) =$

Birth processes and related differential equations

$P_n(t)$ satisfies the following system of differentian eqs. with initial conditions

$$(*)\begin{cases} P_0'(t) = \\ P_1'(t) = \\ P_2'(t) = \\ \vdots \\ P_n'(t) = \\ \vdots \end{cases} \qquad \begin{array}{l} P_0(0) = \\ P_1(0) = \\ P_2(0) = \\ \vdots \\ P_n(0) = \\ \vdots \end{array}$$

Solving this system gives the p.m.f. of X_t for any t

Solving the system of differential equations (*)

$$(*)\begin{cases} P_0'(t) = -\lambda_0 P_0(t), & P_0(0) = 1 \\ P_n'(t) = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t), & P_n(0) = 0 \quad \text{for } n \geq 1 \end{cases}$$

$P_0(t)$:

$$P_0'(t) =$$

$$\frac{P_0'(t)}{P_0(t)} =$$

$$g'(t) =$$

$$g(t) =$$

Solving the system of differential equations $(*)$

$P_n(t)\ ,\ n \geq 1$

Consider the function $\quad Q_n(t) =$

$\left(Q_n(t)\right)' =$

$Q_n(t) =$

$\hookrightarrow\ P_n(t) =$ $\leftarrow$ apply recursively

$$P_1(t) = e^{-\lambda_1 t} \int_0^t \lambda_0 e^{\lambda_1 s} e^{-\lambda_0 s} ds = e^{-\lambda_1 t} \int_0^t \lambda_0 e^{(\lambda_1 - \lambda_0)s} ds \qquad (\text{if } \lambda_1 \neq \lambda_0)$$

$$= e^{-\lambda_1 t} \frac{\lambda_0}{\lambda_1 - \lambda_0} \left(e^{(\lambda_1 - \lambda_0)t} - 1\right) = \frac{\lambda_0}{\lambda_1 - \lambda_0} e^{-\lambda_0 t} - \frac{\lambda_0}{\lambda_1 - \lambda_0} e^{-\lambda_1 t}$$

General solution to (*)

Assume that $\lambda_i \neq \lambda_j$ for $i \neq j$.

Then for $n \geq 1$

$$P_n(t) = \lambda_0 \cdots \lambda_{n-1} \left(B_{0n} e^{-\lambda_0 t} + \cdots + B_{nn} e^{-\lambda_n t} \right)$$

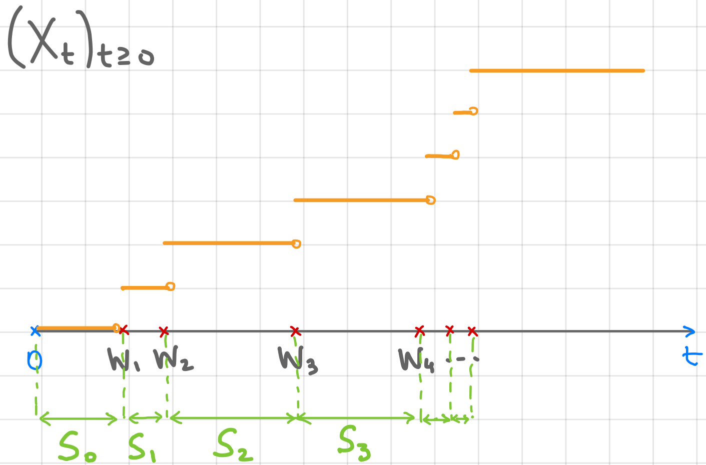
$$B_{kn} =$$

$P_1(t) =$

$P_2(t) =$

$\vdots$

Description of the birth processes via sojourn times



W_i - i-th "birth time"

S_i - "time between (i-1)-th birth and i-th birth"

↳ sojourn times

$$W_i = \sum_{l=0}^{i-1} S_l$$

Alternative way of characterizing $(X_t)_{t\geq 0}$:

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Description of the birth processes via sojourn times

Theorem

Let $(\lambda_k)_{k\geq 0}$ be a sequence of positive numbers. Let $(X_t)_{t\geq 0}$ be a non-decreasing right-continuous process, $X_0 = 0$, taking values in $\{0,1,2\dots\}$. Let $(S_i)_{i\geq 0}$ be the sojourn times associated with $(X_t)_{t\geq 0}$, and define $W_\ell = \sum_{i=0}^{\ell-1} S_i$.

Then conditions

(a)

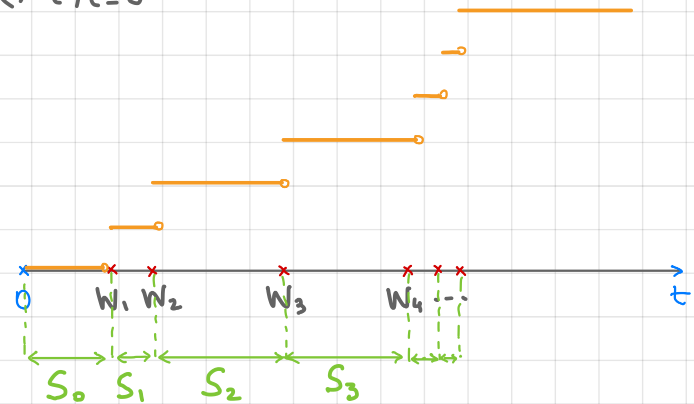
(b)

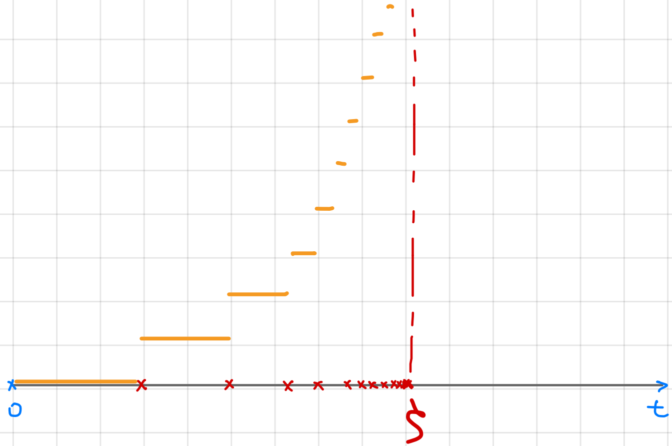
are equivalent to

(c)

Explosion

$(X_t)_{t\geq 0}$





explosion time

population becomes infinite in finite time

Thm. Let $(X_t)_{t\geq 0}$ be a pure birth process of rates $(\lambda_k)_{k\geq 0}$.

Then