MATH180C: Introduction to Stochastic Processes II

Lecture A00: math-old.ucsd.edu/~ynemish/teaching/180cA
Lecture B00: math-old.ucsd.edu/~ynemish/teaching/180cB

Today: Birth and death processes.

Next: PK 6.5

Week 1:

- visit course web site
- homework 0 (due Friday April 1)
- join Piazza

Birth processes and related differential equations

$P_n(t)$ satisfies the following system of differentian eqs. with initial conditions

$$(*)\quad \begin{cases} P_0'(t) = -\lambda_0 P_0(t) \\ P_1'(t) = -\lambda_1 P_1(t) + \lambda_0 P_0(t) \\ P_2'(t) = -\lambda_2 P_2(t) + \lambda_1 P_1(t) \\ \vdots \\ P_n'(t) = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t) \\ \vdots \end{cases}$$

$$\begin{aligned} &P_0(0) = 1 \\ &P_1(0) = 0 = \mathbb{P}(X_0 = 1) \\ &P_2(0) = 0 = P(X_0 = 2) \\ &\vdots \\ &P_n(0) = 0 \\ &\vdots \end{aligned}$$

Solving this system gives the p.m.f. of X_t for any t

$$P(X_t = k) = P_k(t)$$

Solving the system of differential equations (*)

$$(*)\quad \begin{cases} P_0'(t) = -\lambda_0 P_0(t), & P_0(0) = 1 \\ P_n'(t) = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t), & P_n(0) = 0 \quad \text{for } n \geq 1 \end{cases}$$

$P_0(t)$:

$$P_0'(t) = -\lambda_0 P_0(t)$$

$$\frac{P_0'(t)}{P_0(t)} = -\lambda_0$$

$$\Big(\underbrace{\log P_0(t)}_{g(t)}\Big)' = \frac{1}{P_0(t)} \cdot P_0'(t)$$

$$g'(t) = -\lambda_0$$

$$g(t) = -\lambda_0 t + K = \log P_0(t)$$

$$P_0(t) = e^{-\lambda_0 t} \cdot e^{K} = e^{-\lambda_0 t} \cdot C, \quad C > 0$$

$$P_0(0) = 1 = C$$

$$P_0(t) = e^{-\lambda_0 t}$$

$$\parallel$$

$$P(X_t = 0)$$

Solving the system of differential equations (*)

$P_n(t)$, $n \geq 1$

Consider the function $Q_n(t) = e^{\lambda_n t} P_n(t)$

$$(Q_n(t))' = (e^{\lambda_n t} P_n(t))' = \lambda_n e^{\lambda_n t} P_n(t) + e^{\lambda_n t} P_n'(t)$$

$$= \lambda_n e^{\lambda_n t} P_n(t) + e^{\lambda_n t}(-\lambda_n P_n(t) + \lambda_{n-1} e^{\lambda_n t} P_{n-1}(t))$$

$$= \lambda_{n-1} e^{\lambda_n t} P_{n-1}(t)$$

$$Q_n(t) = \int_0^t \lambda_{n-1} e^{\lambda_n s} P_{n-1}(s)\, ds$$

$$\hookrightarrow \quad P_n(t) = e^{-\lambda_n t} \int_0^t \lambda_{n-1} e^{\lambda_n s} P_{n-1}(s)\, ds \leftarrow \text{apply recursively}$$

$$P_1(t) = e^{-\lambda_1 t} \int_0^t \lambda_0 e^{\lambda_1 s} e^{-\lambda_0 s}\, ds = e^{-\lambda_1 t} \int_0^t \lambda_0 e^{(\lambda_1 - \lambda_0)s}\, ds \qquad (\text{if } \lambda_1 \neq \lambda_0)$$

$$= e^{-\lambda_1 t} \frac{\lambda_0}{\lambda_1 - \lambda_0} \left(e^{(\lambda_1 - \lambda_0)t} - 1\right) = \frac{\lambda_0}{\lambda_1 - \lambda_0} e^{-\lambda_0 t} - \frac{\lambda_0}{\lambda_1 - \lambda_0} e^{-\lambda_1 t}$$

General solution to (*)

Assume that $\lambda_i \neq \lambda_j$ for $i \neq j$.

Then for $n \geq 1$

$$P_n(t) = \lambda_0 \cdots \lambda_{n-1} \left(B_{0n} e^{-\lambda_0 t} + \cdots + B_{nn} e^{-\lambda_n t} \right)$$

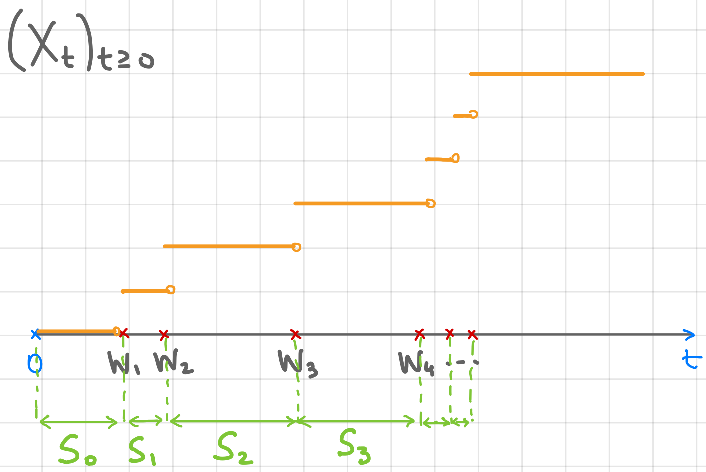
$$B_{kn} = \prod_{\substack{l=0 \\ l \neq k}}^{n} \frac{1}{\lambda_l - \lambda_k}$$

$$P_1(t) = \lambda_0 \left(\frac{1}{\lambda_1 - \lambda_0} e^{-\lambda_0 t} + \frac{1}{\lambda_0 - \lambda_1} e^{-\lambda_1 t} \right)$$

$$P_2(t) = \lambda_0 \lambda_1 \left(\frac{1}{\lambda_1 - \lambda_0} \frac{1}{\lambda_2 - \lambda_0} e^{-\lambda_0 t} + \frac{1}{\lambda_0 - \lambda_1} \frac{1}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} + \frac{1}{\lambda_0 - \lambda_2} \frac{1}{\lambda_1 - \lambda_2} e^{-\lambda_2 t} \right)$$

$\vdots$

Description of the birth processes via sojourn times



W_i – i-th "birth time" S_i – "time between (i-1)-th birth and i-th birth"
↳ sojourn times

$$W_i = \sum_{l=0}^{i-1} S_l$$

Alternative way of characterizing $(X_t)_{t\geq 0}$:

- describe the distribution of $(S_i)_{i\geq 0}$
- describe the jumps $X_{W_{i+1}} - X_{W_i}$

Description of the birth processes via sojourn times

Theorem

Let $(\lambda_k)_{k\geq 0}$ be a sequence of positive numbers. Let $(X_t)_{t\geq 0}$ be a non-decreasing right-continuous process, $X_0=0$, taking values in $\{0,1,2\dots\}$. Let $(S_i)_{i\geq 0}$ be the sojourn times associated with $(X_t)_{t\geq 0}$, and define $W_\ell = \sum_{i=0}^{\ell-1} S_i$.

Then conditions

(a) $S_0, S_1, S_2, \dots$ are independent exponential r.v.s of rates $\lambda_0, \lambda_1, \lambda_2, \dots$

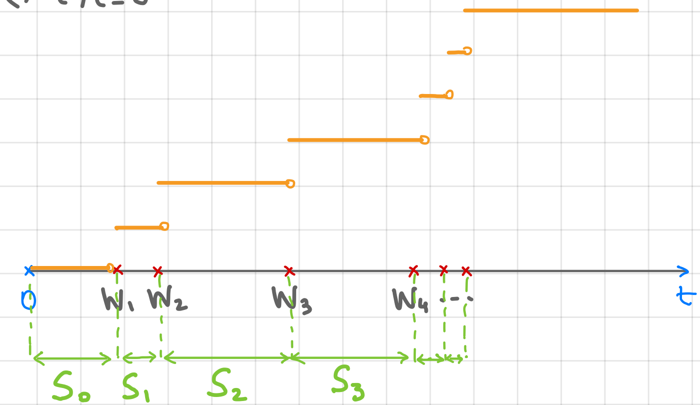
(b) $X_{W_i} = i$

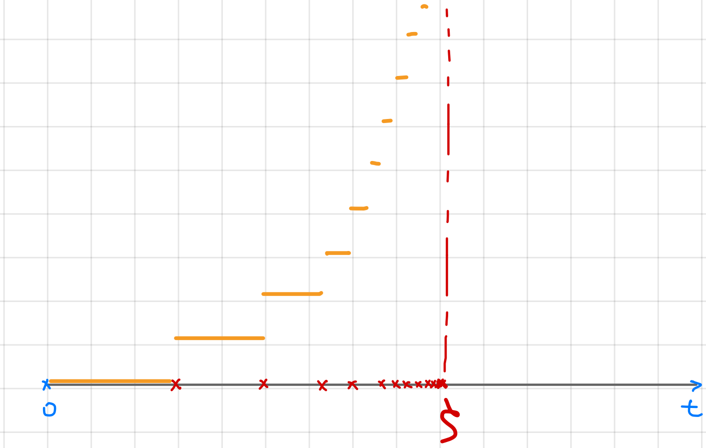
are equivalent to

(c) $(X_t)_{t\geq 0}$ is a pure birth process with paremeters (λ_k)

Explosion

$(X_t)_{t \geq 0}$





explosion time

population becomes infinite in finite time

Thm. Let $(X_t)_{t \geq 0}$ be a pure birth process of rates $(\lambda_k)_{k \geq 0}$.

Then

- if $\displaystyle\sum_{k=0}^{\infty} \frac{1}{\lambda_k} < \infty$, then $P\big((X_t) \text{ explodes}\big) = 1$
- if $\displaystyle\sum_{k=0}^{\infty} \frac{1}{\lambda_k} = \infty$, then $P\big((X_t) \text{ does not explode}\big) = 1$

Hint. $E\left[\displaystyle\sum_{k=0}^{\infty} S_k\right] = \displaystyle\sum_{k=0}^{\infty} \frac{1}{\lambda_k}$