

MATH180C: Introduction to Stochastic Processes II

[Lecture A00: math-old.ucsd.edu/~ynemish/teaching/180cA](http://math-old.ucsd.edu/~ynemish/teaching/180cA)

[Lecture B00: math-old.ucsd.edu/~ynemish/teaching/180cB](http://math-old.ucsd.edu/~ynemish/teaching/180cB)

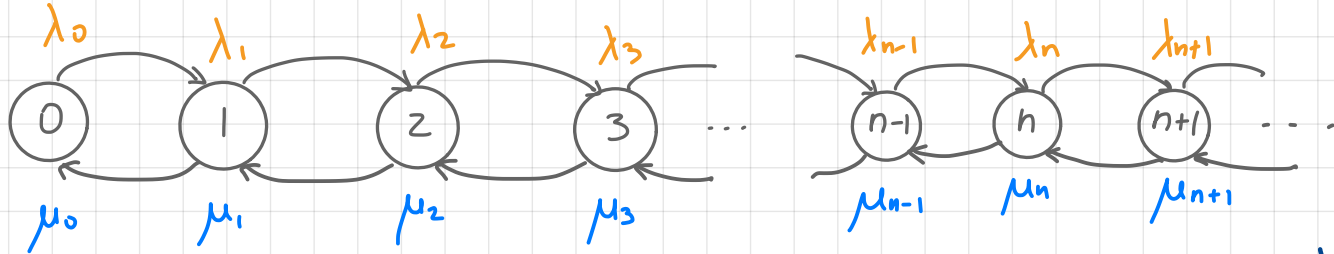
Today: Birth and death processes.
Absorption times.
General CTMC. Matrix
exponentials

Next: PK 6.6, Durrett 4.1

Week 2:

- homework 1 (due Friday April 8)

Alternative (jump and hold) characterization



Sojourn times S_k are independent,

$$\frac{\lambda}{\lambda + \mu} = \frac{\mu}{\lambda + \mu} = \frac{\lambda'}{\lambda' + \mu'} = \frac{\mu'}{\lambda' + \mu'}$$

$\swarrow \text{Exp}(\lambda)$ $\searrow \text{Exp}(\mu)$

Each transition has two parts

- wait in state i for time $\sim \text{Exp}(\lambda_i + \mu_i)$
- then choose where to go:

go $\rightarrow (i+1)$ with probability $\frac{\lambda_i}{\lambda_i + \mu_i}$

go $\leftarrow (i-1)$ with probability $\frac{\mu_i}{\lambda_i + \mu_i}$

Stopping times

Def (Informal). Let $(X_t)_{t \geq 0}$ be a stochastic process and let $T \geq 0$ be a random variable. We call T a **stopping time** if the event

$$\{T \leq t\}$$

can be determined from the knowledge of the process up to time t (i.e., from $\{X_s : 0 \leq s \leq t\}$)

Examples: Let $(X_t)_{t \geq 0}$ be right-continuous

1. $\min\{t \geq 0 : X_t = i\}$ is a stopping time
2. W_k is a stopping time
3. $\sup\{t \geq 0 : X_t = i\}$ is not a stopping time

Strong Markov property

Theorem (no proof)

Let $(X_t)_{t \geq 0}$ be a MC, let T be a stopping time of $(X_t)_{t \geq 0}$. Then, conditional on $T < \infty$ and $X_T = i$,

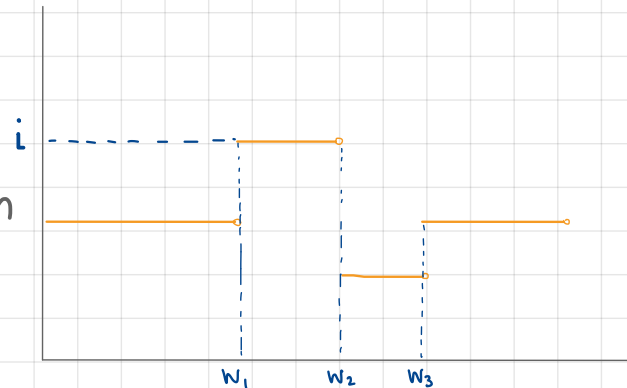
$$(X_{T+t})_{t \geq 0}$$

(i) is independent of $\{X_s, 0 \leq s \leq T\}$

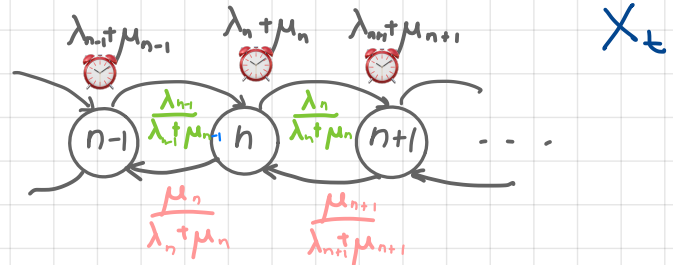
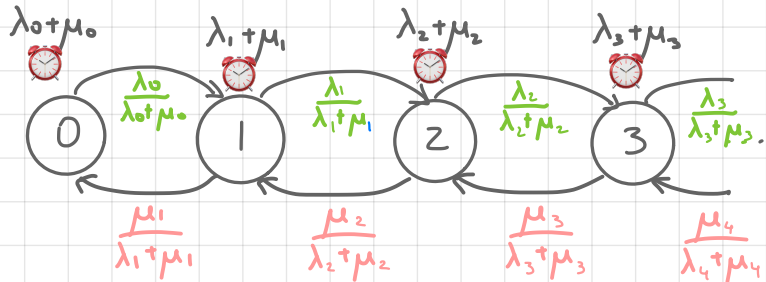
(ii) has the same distribution as $(X_t)_{t \geq 0}$ starting from i .

Example

$(X_{W_i+t})_{t \geq 0}$ has the same distribution as $(X_t)_{t \geq 0}$ conditioned on $X_0 = i$ and is indep. of what happened before

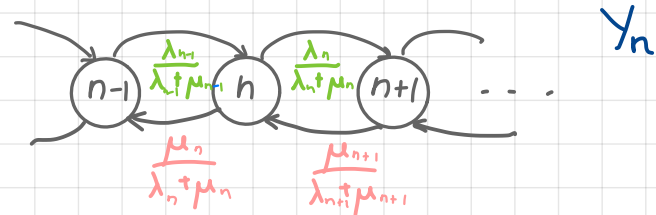
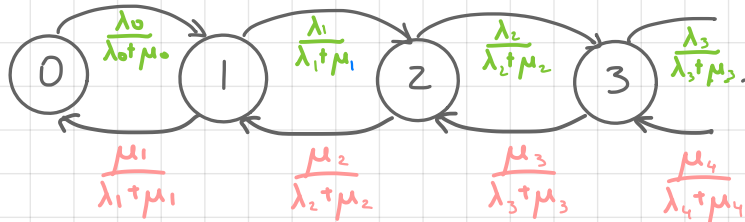


Related discrete time MC.



X_t

Def. Let $(X_t)_{t \geq 0}$ be a continuous time MC, let $W_n, n \geq 0$, be the corresponding waiting (arrival, jump) times. Then we call $(Y_n)_{n \geq 0}$ defined by the jump chain of $(X_t)_{t \geq 0}$.



Y_n

random walk

Mean time until absorption

Let $(X_t)_{t \geq 0}$ be a birth and death process. Denote

$T = \min\{t \geq 0 : X_t = 0\}$ absorption time and

Let $(Y_n)_{n \geq 0}$ be the

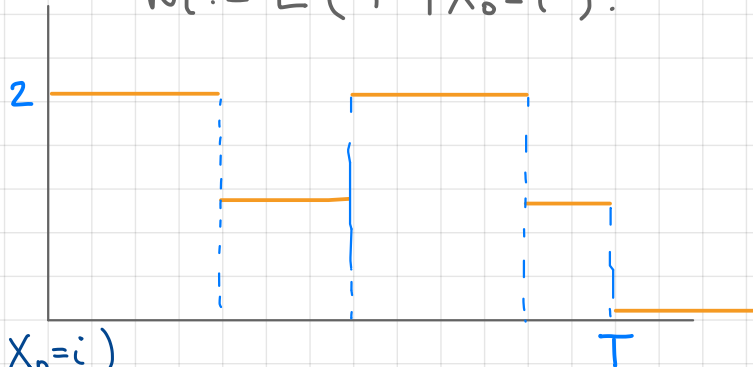
$$w_i := E(T | X_0 = i).$$

jump chain for $(X_t)_{t \geq 0}$.

$$N := \min\{n \geq 0 : Y_n = 0\}$$

$$\text{Then } T = \sum_{k=0}^{N-1} S_k$$

$$= E(S_0 | X_0 = i)$$



$$w_i = E\left(\sum_{k=0}^{N-1} S_k | X_0 = i\right) = \frac{1}{\lambda_i + \mu_i} + E\left(\sum_{k=1}^{N-1} S_k | X_0 = i\right)$$

$$= \frac{1}{\lambda_i + \mu_i} + \underbrace{E\left(\sum_{k=1}^{N-1} S_k | X_0 = i, Y_1 = i+1\right)}_{\substack{\text{|| SMP} \\ w_{i+1}}} P(Y_1 = i+1 | Y_0 = i) \overset{w_{i-1}}{\text{|| SMP}} + E\left(\sum_{k=1}^{N-1} S_k | X_0 = i, Y_1 = i-1\right) P(Y_1 = i-1 | Y_0 = i)$$

Mean time until absorption

$$\begin{cases} w_i = \frac{1}{\lambda_i + \mu_i} + \frac{\lambda_i}{\lambda_i + \mu_i} w_{i+1} + \frac{\mu_i}{\lambda_i + \mu_i} w_{i-1}, \\ w_0 = 0 \end{cases}$$

Alternatively, $y_{i,1}$ and one can show that

$$E(T | X_0 = i) = E\left(\sum_{k=0}^{N-1} \frac{1}{\lambda_{Y_k} + \mu_{Y_k}} \mid Y_0 = i\right)$$

Now apply the first step analysis for the general MC

$$w_i = E\left(\sum_{k=0}^{N-1} g(Y_k) \mid Y_0 = i\right),$$

which leads to (the same) system of equations

$$w_i = g(i) + \sum_{j=1}^{\infty} P_{ij} w_j$$

First step analysis for birth and death processes

Summary:

Let $(X_t)_{t \geq 0}$ be a birth and death process of rates $((\lambda_i, \mu_i))_{i \geq 0}$ with $\lambda_0 = 0$ (state 0 absorbing).

Denote $T = \min\{t : X_t = 0\}$, $u_i = P(X_t \text{ gets absorbed in } 0 | X_0 = i)$

$w_i = E(T | X_0 = i)$ and $p_j = \frac{\mu_1 \mu_2 \dots \mu_j}{\lambda_1 \lambda_2 \dots \lambda_j}$. Then

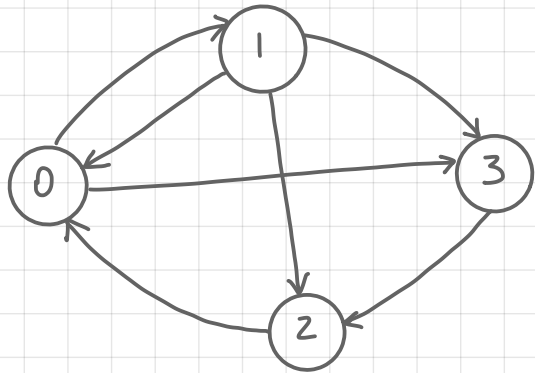
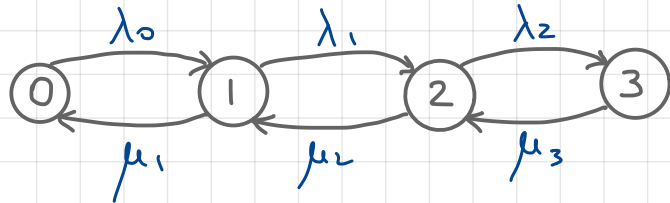
$$u_i = \begin{cases} \frac{\sum_{j=i}^{\infty} p_j}{1 + \sum_{j=i}^{\infty} p_j}, & \text{if } \sum_{j=1}^{\infty} p_j < \infty \\ 1, & \text{if } \sum_{j=1}^{\infty} p_j = \infty \end{cases}$$
$$w_i = \begin{cases} \sum_{j=1}^{\infty} \frac{1}{\lambda_j p_j} + \sum_{k=1}^{i-1} p_k \sum_{j=k+1}^{\infty} \frac{1}{\lambda_j p_j}, & \text{if } \sum_{j=1}^{\infty} \frac{1}{\lambda_j p_j} < \infty \\ \infty, & \text{if } \sum_{j=1}^{\infty} \frac{1}{\lambda_j p_j} = \infty \end{cases}$$

Birth and death processes. Results

- infinitesimal transition probability description
- sojourn time description (jump and hold)
sojourn times are independent exponential r.v.s
$$P(i \rightarrow i+1) = \frac{\lambda_i}{\lambda_i + \mu_i}, \quad P(i \rightarrow i-1) = \frac{\mu_i}{\lambda_i + \mu_i}$$
- system of differential equations for pure birth/death
e.g. $P_i'(t) = -\lambda_i P_i(t) + \lambda_{i-1} P_{i-1}(t)$
- distributions of X_t for linear birth (geometric) and linear death (binomial) processes
- first step analysis giving absorption probabilities and mean time to absorption
- explosion, Strong Markov property etc.

General continuous time MC

Assume for simplicity that the **state space is finite**
 $(X_t)_{t \geq 0}$ is right-continuous



birth and death process

$$P_{i,i+1}(h) = \lambda_i h + o(h)$$

$$P_{i,i}(h) = 1 - \lambda_i h + o(h) \quad h \rightarrow 0$$

general MC

$$P(X_{t+s} = j | X_t = i) = P(X_s = j | X_0 = i)$$

$$P_{ij}(h) = q_{ij} h + o(h)$$

How to define? How to analyze?

Q-matrices (infinitesimal generators)

Let $S = \{0, 1, \dots, N\}$. We call $Q = (q_{ij})_{i,j=0}^N$ a Q-matrix if Q satisfies the following conditions:

(a) $0 \leq -q_{ii} < \infty$ for all i

$$q_i := \sum_{j \neq i} q_{ij}$$

(b) $q_{ij} \geq 0$ for all $i \neq j$

(c) $\sum_j q_{ij} = 0$ for all i

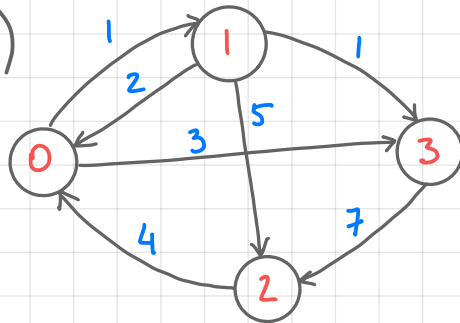
then $q_{ii} = -q_i$

Examples

(a)

$$Q = \begin{pmatrix} -2 & 1 & 1 \\ 2 & -7 & 5 \\ 0 & 2 & -2 \end{pmatrix}$$

(b)



$$\begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} -4 & 1 & 0 & 3 \\ 2 & -8 & 5 & 1 \\ 4 & 0 & -4 & 0 \\ 0 & 0 & 7 & -7 \end{pmatrix} \end{matrix}$$

Matrix exponentials

Let $Q = (q_{ij})_{i,j=1}^N$ be a matrix. Then the series $\sum_{k=0}^{\infty} \frac{Q^k}{k!}$ converges componentwise, and we denote

its sum $\sum_{k=0}^{\infty} \frac{Q^k}{k!} =: e^Q$, the matrix exponential of Q .

In particular, we can define $e^{tQ} = \sum_{k=0}^{\infty} \frac{Q^k t^k}{k!}$ for $t \geq 0$.

Thm. Define $P(t) = e^{tQ}$. Then

(i) for all s, t

(ii) $(P(t))_{t \geq 0}$ is the unique solution to the equations

$$\begin{cases} \frac{d}{dt} P(t) = & , \text{ and } \\ P(0) = . \end{cases} \quad \begin{cases} \frac{d}{dt} P(t) = \\ P(0) = \end{cases}$$