Name (last, first): ______________________________

Student ID: ______________________________

☐ **Write your name and PID on the top of EVERY PAGE.**

☐ **Write the solutions to each problem on separate pages. CLEARLY INDICATE on the top of each page the number of the corresponding problem. Different parts of the same problem can be written on the same page (for example, part (a) and part (b)).**

☐ **Remember this exam is graded by a human being. Write your solutions NEATLY AND COHERENTLY, or they risk not receiving full credit.**

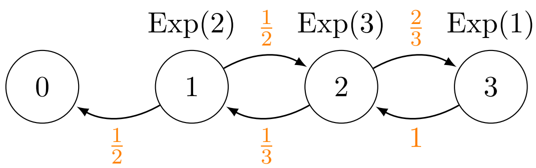
☐ **You may assume that all transition probability functions are STATIONARY.**

☐ **You are allowed to use one 8.5 by 11 inch sheet of paper with handwritten notes (on both sides); no other notes (or books) are allowed.**

1. (30 points) Let $(X_t)_{t\geq 0}$ be a birth and death process on states $\{0,1,2,3\}$ with state 0 absorbing, birth rates $\lambda_1 = 1$, $\lambda_2 = 2$ and the death rates $\mu_1 = 1$, $\mu_2 = 1$, $\mu_3 = 1$.
   (a) (15 points) Draw the diagram of the jump chain of $(X_t)_{t\geq 0}$ and indicate the distribution of the sojourn times.
   (b) (15 points) Suppose that X_0, the state of the process at time $t = 0$, is uniformly distributed on the set $\{1,2,3\}$. Compute the expectation of the time at which the process is absorbed at state 0.

Solution.

(a) The diagram of the jump chain of $(X_t)_{t\geq 0}$ has the following form



The probability $P_{i,i+1}$ (the probability of jumping from state i to state $i+1$) is equal to

$$P_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i},$$

and, similarly,

$$P_{i,i-1} = \frac{\mu_i}{\lambda_i + \mu_i}.$$

The sojourn time at state i has exponential distribution with rate $\lambda_i + \mu_i$.

(b) Denote by v_i the expected time to absorption given that $X_0 = i$, $i \in \{1,2,3\}$. Then, using the first step analysis, v_1, v_2, v_3 satisfy the following system of equations

$$\begin{aligned} v_1 &= \frac{1}{2} + \frac{1}{2}v_2, \\ v_2 &= \frac{1}{3} + \frac{1}{3}v_1 + \frac{2}{3}v_3, \\ v_3 &= 1 + v_2. \end{aligned}$$

Substituting the first and the third equations into the second, we get

$$\begin{aligned} v_2 &= \frac{1}{3} + \frac{1}{3}\left(\frac{1}{2} + \frac{1}{2}v_2\right) + \frac{2}{3}(1 + v_2), \\ v_2 &= \frac{7}{6} + \frac{5}{6}v_2, \\ v_2 &= 7, \quad v_3 = 8, \quad v_1 = 4. \end{aligned}$$

Using the law of total probability, the average time to absorption at state 0 is equal to

$$\frac{1}{3}v_1 + \frac{1}{3}v_2 + \frac{1}{3}v_3 = \frac{1}{3}(7 + 8 + 4) = \frac{19}{3}.$$

2. (40 points) Suppose that the number of printers working at a printing facility at a given time t is given by a continuous time Markov chain $(X_t)_{t\geq 0}$ on the state space $\{0,1,2\}$ with generator

$$Q = \left(\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & -7 & 3 & 4 \\ 1 & 1 & -5 & 4 \\ 2 & 2 & 6 & -8 \end{array}\right).$$

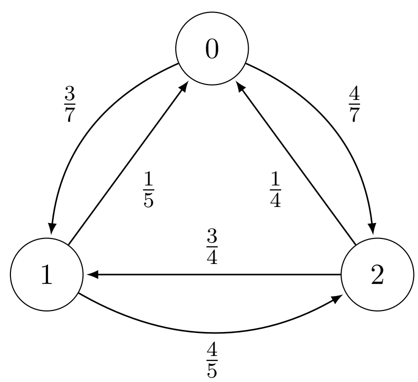
   (a) (10 points) Draw the diagram for the jump chain of $(X_t)_{t\geq 0}$ and explain why $(X_t)_{t\geq 0}$ is irreducible.

   (b) (15 points) Compute the stationary distribution of $(X_t)_{t\geq 0}$. [**Hint.** Remember that there are different ways of finding the stationary distribution].

   (c) (15 points) Suppose that the printing facility works on a 24/7 basis and each printer can produce 100 pages per minute. How many pages does the facility produce on average per minute in the long run?

Solution.

(a) The diagram of the jump chain has the following form



We see that with positive probability the jump chain can transition from any state i to any other state j in one step. Thus, the embedded jump chain is irreducible, and therefore the continuous-time Markov chain $(X_t)_{t\geq 0}$ is irreducible as well.

(b) To compute the stationary distribution $\pi = (\pi_0, \pi_1, \pi_2)$, write the *detailed balance* equation

$$\begin{aligned} 3\pi_0 &= \pi_1, \\ 4\pi_0 &= 2\pi_2, \\ 4\pi_1 &= 6\pi_2. \end{aligned}$$

The first two equations give

$$\pi_1 = 3\pi_0, \quad \pi_2 = 2\pi_0,$$

which make the third equation redundant. Substituting the above into $\sum_{i=0}^{2} \pi_i = 1$ we get

$$\pi_0 + 3\pi_0 + 2\pi_0 = 6\pi_0 = 1.$$

We conclude that

$$\pi_0 = \frac{1}{6}, \quad \pi_1 = \frac{1}{2}, \quad \pi_2 = \frac{1}{3}.$$

Since $\pi = (\frac{1}{6}, \frac{1}{2}, \frac{1}{3})$ is a solution to the detailed balance equation with strictly positive components, π defines the stationary distribution for the Markov chain $(X_t)_{t\geq 0}$.

(c) In the long run, the entries of the stationary distribution give the average amount of time spent by the Markov chain in each of the states. This means, that in the long run, $\frac{1}{6}$ of the time there are 0 printers working, $\frac{1}{2}$ of the time there is only one printer working, and $\frac{1}{3}$ of the time both printers are working. Therefore, on average in the long run the printing facility produces

$$\frac{1}{6} \cdot 0 + \frac{1}{2} \cdot 100 + \frac{1}{3} \cdot 200 = \frac{350}{3}$$

pages per minute.

3. (30 points) Let $(X_t)_{t\geq 0}$ be a continuous-time Markov chain on the state space $\{0, 1, 2\}$ with transition probability functions

$$P(t) = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c|c|c} 0 & 1 & 2 \\ \frac{1}{6} + \frac{3}{2}e^{-4t} - \frac{2}{3}e^{-3t} & \frac{1}{6} - \frac{3}{2}e^{-4t} + \frac{4}{3}e^{-3t} & \frac{2}{3} - \frac{2}{3}e^{-3t} \\ \frac{1}{6} + \frac{1}{2}e^{-4t} - \frac{2}{3}e^{-3t} & \frac{1}{6} - \frac{1}{2}e^{-4t} + \frac{4}{3}e^{-3t} & \frac{2}{3} - \frac{2}{3}e^{-3t} \\ \frac{1}{6} - \frac{1}{2}e^{-4t} + \frac{1}{3}e^{-3t} & \frac{1}{6} + \frac{1}{2}e^{-4t} - \frac{2}{3}e^{-3t} & \frac{2}{3} + \frac{1}{3}e^{-3t} \end{array}.$$

(a) (10 points) Determine the distribution of the sojourn times of the process at states 0, 1 and 2.

(b) (10 points) In the long run, what fraction of time will the process $(X_t)_{t\geq 0}$ spend in state 0? [**Hint.** You can answer this question without solving any equations, and if you do so you should clearly state which results you use.]

(c) (10 points) Let $Q = (q_{ij})_{i,j=0}^{2}$ be the generator matrix of $(X_t)_{t\geq 0}$. Compute q_{10}. Suppose you observe the process jumping from state 2 to state 0. What is the average time that you have to wait until the next time you observe the jump from state 2 to state 0?

Solution.

(a) The distribution of the sojourn times can be read off from the infinitesimal generator Q, and from the relation between the Markov semigroup $P(t)$ and Q we have that $Q = P'(0)$. Therefore, to determine the distribution of the sojourn times it is enough to compute the derivatives of the diagonal entries of $P(t)$ at $t = 0$

$$P'_{00}(0) = -4, \quad P'_{11}(0) = -2, \quad P'_{22}(0) = -1.$$

Thus, the sojourn times at states 0, 1 and 2 have exponential distributions with rates $q_0 = 4$, $q_1 = 2$, $q_2 = 1$ correspondingly.

(b) Let $\pi = (\pi_0, \pi_1, \pi_2)$ be the stationary distribution for the Markov chain $(X_t)_{t\geq 0}$. Then π_i, $i \in \{0, 1, 2\}$, gives the average long run fraction of time spent by the process in state i.

In order to compute π_0, note that from the theorem about the long run behavior of continuous time Markov chains, $P_{i0} \to \pi_0$ as $t \to \infty$. If we take the limit in the above explicit formula for $P(t)$ we get

$$\lim_{t\to\infty} P(t) = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \\ \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \\ \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \end{pmatrix}, \tag{1}$$

and thus on average in the long run the process spends 1/6 of the time in state 0.

(c) If Q is the infinitesimal generator of $(X_t)_{t\geq 0}$, then

$$q_{10} = P'_{10}(0) = 0.$$

In particular this means that the process cannot jump directly from state 1 to state 0; the process can jump to state 0 only from state 2.

In order to compute the average time required to observe the transition from 2 to 0 happening again, we can either apply the first step analysis, or use the theorem about the long run behavior of the continuous time Markov chains. I present below the second solution.

From the theorem about the long run behavior of the continuous time Markov chains,

$$\pi_i = \frac{1}{q_i m_i},$$

where m_i is the average return time to state i. From this we have that the average return time to 0 is given by

$$m_0 = \frac{1}{q_0 \pi_0} = \frac{1}{4 \cdot \frac{1}{6}} = \frac{3}{2}.$$

If you observe the transition from state 2 to state 0, then the return of the process to state 0 can only occur through a transition from 2 to 0 ($q_{10} = 0$, so the jumps from 1 to 0 are not allowed). Therefore, the average time to see again the transition from 2 to 0 is equal to $m_0 = \frac{3}{2}$.