Name (last, first): ______________________

Student ID: ______________________

☐ **Write your name and PID on the top of EVERY PAGE.**

☐ **Write the solutions to each problem on separate pages. CLEARLY INDICATE on the top of each page the number of the corresponding problem. Different parts of the same problem can be written on the same page (for example, part (a) and part (b)).**

☐ **Remember this exam is graded by a human being. Write your solutions NEATLY AND COHERENTLY, or they risk not receiving full credit.**

☐ **You are allowed to use one 8.5 by 11 inch sheet of paper with handwritten notes (on both sides); no other notes (or books) are allowed.**

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1. (40 points) Let $Y \sim \text{Exp}(\lambda)$ for some $\lambda > 0$ and let $X \sim \text{Exp}(\frac{1}{Y})$.
   (a) Compute $E(X)$.
   (b) Show that if T is a random variable having exponential distribution with rate (i.e., if $T \sim \text{Exp}(\mu)$), then
   $$E(T^2) = \frac{2}{\mu^2}.$$
   (c) Use part (b) to compute $E(X^2)$.
   (d) Compute the variance $\text{Var}(X)$ of the random variable X.

Solution.

(a) Condition of the value of Y and use $E(X \,|\, Y = y) = y$

$$E(X) = \int_0^\infty E(X \,|\, Y = y)\lambda e^{-\lambda y} dy = \int_0^\infty y\lambda e^{-\lambda y} dy = \frac{1}{\lambda}. \tag{1}$$

(b) If $T \sim \text{Exp}(\mu)$ then integration by part gives

$$E(T^2) = \int_0^\infty x^2 \mu e^{-\mu x} dx = -x^2 e^{-\mu x}\Big|_0^\infty + \int_0^\infty 2xe^{-\mu x} dx = \frac{2}{\mu^2}. \tag{2}$$

(c) Now compute $E(X^2)$ using $E(X^2 \,|\, Y = y) = 2y^2$ (follows from (b) with $\mu = 1/y$)

$$E(X^2) = \int_0^\infty 2y^2 \lambda e^{-\lambda y} dy = \frac{4}{\lambda^2}. \tag{3}$$

(d) We can now compute the variance

$$\text{Var}(X) = \frac{4}{\lambda^2} - \frac{1}{\lambda^2} = \frac{3}{\lambda^2}. \tag{4}$$

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

2. (30 points) The time intervals between two consecutive rainstorms in San Diego are independent identically distributed random variables with density (in years)

$$f(x) = \begin{cases} 2(1-x), & x \in (0,1) \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

(a) Compute the long run expected time between the last rainstorm and the next rainstorm.

(b) What is the long run probability that there will be no rainstorms in San Diego in the next 6 months?

Solution.

(a) If $\delta(t)$ is the current life (age) of the renewal process at time t (time from the last rainstorm to time t), and $\gamma(t)$ is the residual life of the renewal process at time t (time until the next rainstorm after time t), then we have to compute

$$\lim_{t\to\infty} E(\delta(t)+\gamma(t)) = \lim_{t\to\infty} E(\beta(t)). \tag{6}$$

Lecture 19, page 3:

$$\lim_{t\to\infty} E(\beta(t)) = \frac{\sigma^2+\mu^2}{\mu}, \tag{7}$$

where μ and σ^2 are the mean and variance of the interrenewal times.

$$\mu = \int_0^1 2x(1-x)dx = (x^2 - \frac{2x^3}{3})\Big|_0^1 = \frac{1}{3}, \tag{8}$$

$$\mu^2+\sigma^2 = \int_0^1 2x^2(1-x)dx = \left(\frac{2x^3}{3} - \frac{x^4}{2}\right)\Big|_0^1 = \frac{1}{6}, \tag{9}$$

therefore

$$\lim_{t\to\infty} E(\beta(t)) = \frac{1}{2}. \tag{10}$$

(b) In terms of the renewal process, the long run probability that there will be no rainstorm in the next 6 months is given by

$$\lim_{t\to\infty} P(\gamma(t) > 0.5). \tag{11}$$

Lecture 17, page 4:

$$\lim_{t\to\infty} P(\gamma(t) > 0.5) = \int_{0.5}^{\infty} \frac{1}{\mu}(1-F(x))dx, \tag{12}$$

where $F(x)$ is the interrenewal distribution. Note, that $F(x) = 1$ for $x \geq 1$. For $x \in (0,1)$

$$F(x) = \int_0^x 2(1-s)ds = -(1-s)^2\Big|_0^x = 1-(1-x)^2. \tag{13}$$

Therefore,

$$\lim_{t\to\infty} P(\gamma(t) > 0.5) = \int_{0.5}^1 3(1-x)^2 dx = -(1-x)^3\Big|_{0.5}^1 = \frac{1}{8}. \tag{14}$$

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

3. (30 points) Suppose that certain company is using age replacement policy for replacing lightbulbs in its offices: a lightbulb is replaced either upon its failure, or after reaching age $T > 0$, whichever comes first. Suppose that each bulb replacement costs 1 dollar, but if it happens due to a failure, then it incurs **additional** costs of $\frac{3}{2}$ dollars per replacement. It is given that the lifetime of a lightbulb has a uniform distribution on the interval [0,3].

   Determine the optimal replacement age T (that minimizes the long run mean cost of the replacement) and compute the long run mean replacement cost per unit of time for this choice of T. Compare it to the costs of replacement upon failure.

Solution. Use age replacement strategy from Lecture 17. If the cost of one replacement is K dollars, each replacement due to a failure costs additional c dollars, T is the replacement age and the interrenewal distribution is given by F, then the long run replacement cost (per unit) is given by

$$C(T) = \frac{K + cF(T)}{\int_0^T (1 - F(x))dx}. \tag{15}$$

In our particular case, $K = 1$, $c = 3/2$ and

$$F(t) = \begin{cases} 0, & t \leq 0, \\ t/3, & 0 < t \leq 3, \\ 1, & t > 3, \end{cases} \tag{16}$$

so

$$\int_0^T (1 - F(x))dx = T - \frac{T^2}{6} \tag{17}$$

for $0 \leq T \leq 3$. Therefore,

$$C(T) = \frac{1 + T/2}{T - T^2/6}. \tag{18}$$

Find the minimum

$$C'(T) = \frac{\frac{1}{2}(T - T^2/6) - (1 + T/2)(1 - T/3)}{(T - T^2/6)^2} = \frac{T^2/12 + T/3 - 1}{(T - T^2/6)^2} = 0. \tag{19}$$

Multiplying the numerator by 12, we get that the equation

$$T^2 + 4T - 12 = 0, \tag{20}$$

has two solutions, $T = -6$ and $T = 2$. Point $T = 2$ is the point of minimum, therefore, the optimal long run replacement cost per unit of time is equal to

$$C(2) = \frac{1 + 1}{2 - 4/6} = \frac{3}{2}. \tag{21}$$

The cost of replacement upon failure is $K + c = 1 + \frac{3}{2} = \frac{5}{2} > \frac{3}{2}$.

The failure rate per unit of time is 2/3 (since the expected length of the interrenewal time is 3/2). Therefore, the long-run replacement cost per unit of time without using the age replacement policy is $\frac{5}{2} \cdot \frac{2}{3} = \frac{5}{3} > \frac{3}{2}$.

Therefore, the age replacement policy with the replacement age $T = 2$ will save the company $\frac{5}{3} - \frac{3}{2} = \frac{1}{6}$ dollars per unit of time in the long run.

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)