Math 285: Winter 2022
Homework 3

Due: Friday, February 4, 11:59 PM

Upload the homework to Gradescope by Friday, February 4, 11:59 PM. Late homework will not be accepted.

1. Let $(X_n)_{n=0}^{\infty}$ be an irreducible, reccurent Markov chain with state space S.

(a) Show that, for all $i, j \in S$, $\mathbb{P}_i(X_n = j \text{ for some } n) = 1$.

(b) Deduce that for all $i, j \in S$, $\mathbb{P}_i(X_n = j \text{ for infinitely many } n) = 1$.

2. Let $(X_n)_{n\geq 0}$ be a Markov chain with state space S and transition probabilities $p(i, j)$. Assume $p(i, i) < 1$ for all $i \in S$. Define a sequence of random times T_k by

$$T_0 = 0, \quad T_k = \min\{n > T_{k-1} \colon X_n \neq X_{T_{k-1}}\}.$$

(I.e. T_k is the kth time that the Markov chain moves to a new state.) Let $Y_n = X_{T_n}$. Show that $(Y_n)_{n\geq 0}$ is a Markov chain with state space S and transition probabilities $q(i, j)$ given by

$$q(i, i) = 0, \quad q(i, j) = \frac{p(i, j)}{1 - p(i, i)} \text{ for } i \neq j.$$

3. Suppose $\mathbf{P}$ is the transtion matrix for a finite-state irreducible Markov chain, with stationary distribution π. Let $\epsilon > 0$, and let $\mathbf{Q} = \epsilon \mathbf{I} + (1 - \epsilon)\mathbf{P}$ where $\mathbf{I}$ is the identity matrix. Show that $\mathbf{Q}$ is the transition matrix for an irreducible *aperiodic* Markov chain with stationary distribution π. (In other words: every irreducible chain is "arbitrarily close" to an irreducible aperiodic chain with the same stationary distribution.)

4. (Lawler, Exercise 2.3) Consider the Markov chain with state space $S = \{0, 1, 2, \ldots\}$ and transition probabilities

$$p(x, x + 1) = 2/3, \quad p(x, 0) = 1/3.$$

Show that the chain is positive recurrent and give the limiting probability π.

5. (Lawler, Exercise 2.16) Let $p_1, p_0, p_{-1}, \ldots$ be a probability distribution on $\{\ldots, -2, -1, 0, 1\}$ with $p_1 \in (0, 1)$ and negative mean

$$\sum_n n p_n = \mu < 0.$$

Define a Markov chain X_n on the nennegative integers with transition probabilities

$$p(n, m) = p_{m-n}, \quad m > 0,$$

$$p(n,0) = \sum_{m \le 0} p_{m-n}.$$

In other words, X_n acts like a random walk with increments given by the p_i except that the walk is forbidden to jump below 0. The purpose of this exercise is to show that the chain is positive recurrent.

(a) Let $\pi(n)$ be an invariant probability for the chain. Show that for each $n > 0$

$$\pi(n) = \sum_{m=n-1}^{\infty} \pi(m) p_{n-m}.$$

(b) Let $q_n = p_{1-n}$. Show there exists an $\alpha \in (0,1)$ such that

$$\alpha = q_0 + q_1\alpha + q_2\alpha^2 + \cdots .$$

[Hint: q_n is the probability distribution of a random variable with mean greater than 1. The right-hand side is the generating function of the q_n.)

(c) Use the α from (b) to find the invariant probability distribution for the chain.