Math 285: Winter 2022
Homework 5

Due: Sunday, February 20, 11:59 PM

Upload the homework to Gradescope by Sunday, February 20, 11:59 PM. Late homework will not be accepted.

1. Consider the following Hidden Markov Model. Let $(Y_n)_{n\geq 0}$ be a Markov chain with state space $S = \{1, 2\}$ and transition probabilities $p(1, 1) = p(2, 2) = 0.8$ and $p(1, 2) = p(2, 1) = 0.2$. We think of these states as representing two coins, where coin 1 is fair (with probability 0.5 of coming up heads H or tails T), and coin 2 is biased with probability 0.7 of H and 0.3 of T. We perform 5 coin tosses, and for $k = 0, 1, 2, 3, 4$ we use coin i for our $(k + 1)$st toss if $Y_k = i$. Assume $Y_0 = 1$, so the fair coin is tossed first.

(a) Using the forward algorithm, compute the probability that the fair coin is tossed all 5 times, given that the results are $THHHT$.

(b) Use the Viterbi algorithm to predict the most likely sequence of coins tossed, given that the results are $THHHT$.

2. Let U_n and V_n be i.i.d. sequences of uniform random variables, independent of each other. Let R and S be countable sets, and let $F\colon S \times [0, 1] \to R$ and $G\colon S \times [0, 1] \to S$ be functions. Let Y_0 be a random variable, and inductively define

$$X_n = F(Y_n, V_n)$$
$$Y_{n+1} = G(Y_n, U_n).$$

Show that $(X_n, Y_n)_{n\geq 0}$ is a Hidden Markov Model, and compute the emission probabilities. Further, show that every Hidden Markov model can be constructed in this manner.

3. (Lawler, Exercise 3.3) Suppose X_t and Y_t are independent Poisson processes with parameters λ_1 and λ_2, respectively, measuring the number of calls arriving at two different phones. Let $Z_t = X_t + Y_t$.

(a) Show that Z_t is a Poisson process. What is the rate parameter for Z?

(b) What is the probability that the first call comes on the first phone?

(c) Let T denote the first time that at least one call has come from each of the phones. Find the density and distribution function of the random variable T.

4. Let $(X_t)_{t\geq 0}$ be a continuous-time Markov chain with state space $S = \{1, 2, \ldots\}$ and transition rates $q(i, i + 1) = i$ for all $i \in S$, and $q(i, j) = 0$ for $j \neq i + 1$. (This process, called a Yule process, models the size of a population in which there are no deaths, and each individual independently reproduces with birth rate 1.)

(a) Let $T = \min\{t\colon X_t = 4\}$. Calculate $\mathbb{E}_1[T]$.

(b) Verify that, for each $t > 0$, the random variable X_t has a geometric distribution with parameter e^{-t}. That is, show that for all positive integers n,

$$\mathbb{P}_1(X_t = n) = e^{-t}(1 - e^{-t})^{n-1}.$$