Math 285: Winter 2022

Homework 6

Due: Friday, March 4, 11:59 PM

Upload the homework to Gradescope by Friday, March 4, 11:59 PM. Late homework will not be accepted.

1. Let $(X_t)_{t\geq 0}$ be a Markov chain, with state space S, transition rates $q(i,j)$ and $q(i) = \sum_{j\neq i} q(i,j) > 0$, and transition kernel $p_t(i,j) = \mathbb{P}_i(X_t = j)$ for $t \geq 0$ and $i,j \in S$.

(a) For a state $i \in S$, show that

$$\int_0^\infty p_t(i,i)\,dt = \frac{1}{q(i)} \sum_{n=0}^{\infty} \tilde{p}_n(i,i)$$

where $\tilde{p}_n(i,j)$ denotes the n-step transition matrix for the jump chain; i.e. $\tilde{p}_n(i,j) = \mathbb{P}_i(X_{J_n} = j)$.

(b) Conclude that, for any state $i \in S$,

$$\begin{aligned} i \text{ is recurrent if } &\quad \int_0^\infty p_t(i,i)\,dt = \infty, \\ i \text{ is transient if } &\quad \int_0^\infty p_t(i,i)\,dt < \infty. \end{aligned}$$

2. Let $(X_t)_{t\geq 0}$ be a Markov chain with state space S, and transition kernel $p_t(i,j)$ for $t \geq 0$ and $i,j \in S$. For given states $i,j \in S$, show that if there exists some time $t > 0$ so that $p_t(i,j) > 0$, then in fact $p_t(i,j) > 0$ *for all* $t > 0$. (I.e. there is never periodicity for continuous-time Markov chains.)

3. Consider a queueing system in which customers arrive according to a Poisson process of rate λ, are served independently with $\mathrm{Exp}(\mu)$ service times, and there are $k \geq 1$ servers (so up to k customers can be served at a given time). This model is called an M/M/k queue.

(a) Model this process as a birth and death chain.

(b) Show that the chain is recurrent if and only if $\lambda \leq k\mu$.

(c) Show that the chain is positive recurrent if and only if $\lambda < k\mu$.

4. Consider a modification of the M/M/1 queue in which, when there are n customers already in the queue, a new potential customer decides to join the queue with probability $r_n \in (0,1)$, and goes away with probability $1 - r_n$. This queue can be modeled by a birth and death chain in which $\lambda_n = r_n\lambda$ for some $\lambda > 0$ and $\mu_n = \mu$ for all $n \geq 1$. Show that, if $\lim_{n\to\infty} r_n = 0$, this chain is positive recurrent for any $\mu, \lambda > 0$.