Math 285: Winter 2022
Homework 7

Due: Friday, March 11, 11:59 PM

Upload the homework to Gradescope by Friday, March 11, 11:59 PM. Late homework will not be accepted.

1. Let $(X_n)_{n\geq 0}$ be a simple random walk on $\mathbb{Z}$, conditioned with $X_0 = 0$. Let $M_n = X_n^3 - 3nX_n$ for $n \geq 0$. Show that $(M_n)_{n\geq 0}$ is a martingale.

2. Let $X_1, X_2, X_3, \ldots$ be independent and identically distributed random variables. Their common *moment generating function* is $\varphi(t) = \mathbb{E}[e^{tX_n}]$ (for any $n \geq 1$). Let $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$ for $n \geq 1$. Fix $t \in \mathbb{R}$, and suppose that $\varphi(t) < \infty$. Define $M_n = \varphi(t)^{-n} e^{tS_n}$ for $n \geq 0$. Show that $(M_n)_{n\geq 0}$ is a martingale.

3. Let $(X_n)_{n\geq 0}$ be a non-symmetric random walk on $\mathbb{Z}$: $(X_n)_{n\geq 0}$ is a Markov chain with state space $\mathbb{Z}$ and transition probabilities $p(i, i+1) = p$ and $p(i, i-1) = 1 - p$ for all $i \in \mathbb{Z}$ and some $p \in (0, 1)$ with $p \neq \frac{1}{2}$. Fix $N > 0$, and condition $X_0 = i$ for some $i \in \{1, \ldots, N-1\}$. As usual let $\tau_k = \min\{n \colon X_n = k\}$.

(a) Let $M_n = [(1-p)/p]^{X_n}$. Show that $(M_n)_{n\geq 0}$ is a martingale.

(b) Use the Optional Sampling Theorem to calculate $\mathbb{P}_i(\tau_N < \tau_0)$.

4. *Jensen's inequality* states that if X is a random variable, $f \colon \mathbb{R} \to \mathbb{R}$ is a convex function, and if $\mathbb{E}[|X|] < \infty$ and $\mathbb{E}[|f(X)|] < \infty$, then $\mathbb{E}[f(X)] \geq f(\mathbb{E}[X])$.

(a) Prove the *conditional Jensen's inequality*: if f and X are as above, and $Y_1, \ldots, Y_n$ are any discrete random variables, then

$$\mathbb{E}[f(X)|Y_1, \ldots, Y_n] \geq f(\mathbb{E}[X|Y_1, \ldots, Y_n]).$$

(b) Use part (a) to establish the following fact: if $(X_n)_{n\geq 0}$ is a martingale, $f \colon \mathbb{R} \to \mathbb{R}$ is convex, and if $Y_n = f(X_n)$ satisfies $\mathbb{E}[|Y_n|] < \infty$ for all n, then $(Y_n)_{n\geq 0}$ is a submartingale.