MATH 285: Stochastic Processes

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Today: Positive and null recurrence

- Homework 2 is due on Friday, January 21 11:59 PM

Positive recurrence and stationary distribution

Def 9.2 Let i be a recurrent state for MC (X_n).
Denote $T_i = \min\{n \geq 1 : X_n = i\}$. If $\mathbb{E}_i[T_i] < \infty$, then we call i positive recurrent. If $\mathbb{E}_i[T_i] = \infty$, the we call i null recurrent.

Prop 9.4 In a finite-state irreducible Markov chain all states are positive recurrent.

Thm 10.2 Let (X_n) be a time homogeneous MC with state space S, and suppose that the chain possesses a stationary distribution π.

(1) If (X_n) is irreducible, then $\pi(j) > 0$ for all $j \in S$

(2) In general, if $\pi(j) > 0$, then j is positive recurrent.

Positive recurrence and stationary distributions

Thm 9.6 Let (X_n) be a Markov chain with a state space that is countable (but not necessarily finite).

Suppose there exists a positive recurrent state $i \in S$, $\mathbb{E}_i[T_i] < \infty$.

For each state $j \in S$ define

$$\gamma(i,j) = \mathbb{E}_i\left[\sum_{n=0}^{T_i - 1} \mathbb{1}_{\{X_n = j\}}\right]$$

(the expected number of visits to j before reaching i).

Then the function $\pi : S \to [0,1]$

$$\pi(j) = \frac{\gamma(i,j)}{\mathbb{E}_i[T_i]}$$

is a stationary distribution for (X_n).

Positive recurrence and stationary distributions

Proof of Thm. 9.6 Recall $T_i = \min\{n \geq 1 : X_n = i\}$.

(i) $\sum_{j \in S} \gamma(i,j) = \mathbb{E}_i[T_i]$

$$\sum_{j \in S} \gamma(i,j) = \sum_{j \in S} \mathbb{E}_i\left[\sum_{n=0}^{T_i - 1} \mathbb{1}_{\{X_n = j\}}\right] = \mathbb{E}_i\left[\sum_{n=0}^{T_i - 1} \underbrace{\sum_{j \in S} \mathbb{1}_{\{X_n = j\}}}_{=1}\right] = \mathbb{E}_i(T_i)$$

(ii) Enough to show that $\forall j$ $\gamma(i,j) = \sum_{k \in S} \gamma(i,k)\, p(k,j)$

Denote $\pi = (\pi(j))_{j \in S}$ with $\pi(j) := \dfrac{\gamma(i,j)}{\mathbb{E}_i[T_i]}$. Then $\forall j \in S$

- $\pi(j) \geq 0$
- $\pi(j) = \sum_{k \in S} \pi(k)\, p(k,j)$
- $\sum_{j \in S} \pi(j) = \sum_{j \in S} \dfrac{\gamma(i,j)}{\mathbb{E}_i[T_i]} = 1$

(iii) $\forall j$ $\gamma(i,j) = \sum_{k \in S} \gamma(k)\, p(k,j)$

- Given that $X_0 = 0$, for any $j \in S$ $\sum_{n=0}^{T_i - 1} \mathbb{1}_{\{X_n = j\}} = \sum_{n=1}^{T_i} \mathbb{1}_{\{X_n = j\}}$

Positive recurrence and stationary distributions

- $$\gamma(i,j) = \mathbb{E}_i\left[\sum_{n=1}^{T_i} \mathbb{1}_{\{X_n=j\}}\right] = \mathbb{E}_i\left[\sum_{n=1}^{\infty} \mathbb{1}_{\{n\le T_i\}} \mathbb{1}_{\{X_n=j\}}\right]$$
$$= \sum_{n=1}^{\infty} \mathbb{E}_i\left[\mathbb{1}_{\{n\le T_i,\, X_n=j\}}\right] = \sum_{n=1}^{\infty} \mathbb{P}_i\left[X_n=j,\ n\le T_i\right]$$

- For any $n\ge 1$ and $j\in S$ $\quad \{n\le T_i\} = \{n-1\ge T_i\}^c$

$$\mathbb{P}_i\left[n\le T_i,\ X_n=j\right] = \sum_{k\in S} \mathbb{P}_i\left[X_n=j,\ n\le T_i,\ X_{n-1}=k\right] \quad \{T_i=k\}$$
$$= \sum_{k\in S} \mathbb{P}_i\left[X_n=j \mid X_{n-1}=k,\ n\le T_i\right] \mathbb{P}_i\left[n\le T_i,\ X_{n-1}=k\right]$$
$$\overset{MP}{=} \sum_{k\in S} p(k,j)\, \mathbb{P}_i\left[n\le T_i,\ X_{n-1}=k\right]$$

- $$\gamma(i,j) = \sum_{k\in S} p(k,j) \sum_{n=1}^{\infty} \mathbb{P}_i\left[X_{n-1}=k,\ n\le T_i\right] \qquad n-1=\ell$$
$$= \sum_{k\in S} p(k,j) \sum_{\ell=0}^{\infty} \mathbb{P}_i\left[X_\ell=k,\ \ell\le T_i-1\right] = \sum_{k\in S} p(k,j)\, \underbrace{\mathbb{E}_i\left[\sum_{\ell=0}^{T_i-1} \mathbb{1}_{\{X_\ell=k\}}\right]}_{=\gamma(i,k)}$$

Positive recurrence and stationary distributions

Corollary 10.1 If i is a positive recurrent state, then the stationary distribution π defined in Thm 9.6 satisfies

$$\pi(i) = \frac{1}{\mathbb{E}_i[T_i]}$$

Proof. Follow from Thm 9.6 and $\gamma(i,i) = 1$. ■

Corollary 11.1 For an irreducible Markov chain, TFAE

(1) there exists a stationary distribution with all entries > 0

T10.2 $\Uparrow$

(2) there exists a stationary distribution

T9.6 $\Uparrow$

(3) there exists a positive recurrent state

$\Uparrow$

(4) all states are positive recurrent

Positive recurrence is a class property!

Proof. By Thm 10.2, $(1) \Rightarrow (4)$

Example: Birth and death chain

Let (X_n) be a birth and death chain: $S = \{0, 1, \dots\}$

- $p(i, i+1) = q$, $p(i, i-1) = 1-q$ for $i \geq 1$
- $p(0,1) = q$, $p(0,0) = 1-q$

$q \in (0,1)$

(X_n) is irreducible

$\beta = \dfrac{q}{1-q}$

Q: Does stationary distribution exist?

$$\begin{cases} \pi(0) = \pi(0)(1-q) + \pi(1)(1-q) \\ \pi(i) = \pi(i-1)q + \pi(i+1)(1-q), \quad i \geq 1 \end{cases}$$

$$\begin{cases} \pi(1) = \beta\pi(0) \\ \pi(i) - \pi(i+1) = \beta(\pi(i-1) - \pi(i)) \end{cases}$$

$$\begin{cases} \pi(0) - \pi(1) = (1-\beta)\pi(0) \\ \pi(i) - \pi(i+1) = \beta^i(1-\beta)\pi(0) \end{cases}$$

$$\pi(0) - \pi(i+1) = (1-\beta^{i+1})\pi(0)$$

$$\pi(i+1) = \beta^{i+1}\pi(0)$$

$\sum_{i=0}^{\infty} \pi(i) = 1$, so stationary distribution exists

$$\Longleftrightarrow \sum_{i=0}^{\infty} \beta^i \pi(0) = 1 \Longleftrightarrow \sum_{i=0}^{\infty} \beta^i < \infty \Longleftrightarrow |\beta| < 1$$

Example: Birth and death chain

$\beta < 1 \iff q < \frac{1}{2}$

- If $q < \frac{1}{2}$, then $\sum_{i=0}^{\infty} \beta^i = \frac{1}{1-\beta}$, $\begin{cases} \pi(0) = 1-\beta \\ \pi(i) = \beta^i (1-\beta) \end{cases}$

  All states are positive recurrent.

- If $q = \frac{1}{2}$, then (X_n) is not positive recurrent.

  (X_n) is recurrent: if $(\tilde{X}_n)$ is a SSRW on $\mathbb{Z}$, then

$$\mathbb{P}_0[\tilde{T}_0 < \infty] = 1 = \mathbb{P}_0[\tilde{T}_0 < \infty \mid \tilde{X}_1 = 1] \cdot \frac{1}{2} + \mathbb{P}_0[\tilde{T}_0 < \infty \mid \tilde{X}_1 = -1] \cdot \frac{1}{2}$$

$$= \mathbb{P}_0[\tilde{T}_0 < \infty \mid \tilde{X}_1 = 1]$$

  At the same time $\mathbb{P}_0[T_0 < \infty] = \frac{1}{2} + \mathbb{P}_0[T_0 < \infty \mid X_1 = 1] \cdot \frac{1}{2}$

  and $\mathbb{P}_0[T_0 < \infty \mid X_1 = 1] = \mathbb{P}_0[\tilde{T}_0 < \infty \mid \tilde{X}_1 = 1] = 1$

  We conclude that (X_n) is null recurrent

- If $q > \frac{1}{2}$, then (X_n) is transient