MATH 285: Stochastic Processes

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Today: Ergodic theorem

- Homework 3 is due on Friday, February 4, 11:59 PM

Probability generating function

Def Let Y be a random variable with values in $\{0,1,2,\dots\}$.
We call the function

$$\varphi_Y(s) := \mathbb{E}[s^Y] = \sum_{k=0}^{\infty} s^k \, \mathbb{P}[Y=k]$$

the probability generating function of Y.

Properties:

(1) $\varphi_Y(s)$ is analytic on $(-1,1)$; $\varphi_Y^{(n)}(0) = n!\, \mathbb{P}[Y=k]$

(2) $\varphi_Y(1) = 1$; $\varphi_Y(0) = \mathbb{P}[Y=0]$

(3) For $|s|<1$, $\varphi_Y'(s) = \sum_{k=1}^{\infty} k s^{k-1} \mathbb{P}[Y=k]$; if $\mathbb{E}[Y]<\infty$, then $\varphi_Y'(1) = \mathbb{E}[Y]$

(4) For $|s|<1$, $\varphi_Y''(s) = \sum_{k=2}^{\infty} k(k-1) s^{k-2} \mathbb{P}[Y=k]$; in particular, if $\mathbb{P}[Y\geq 2] > 0$, then $\varphi_Y(s)$ is (strictly) convex on $(0,1)$

Ergodic Theorem

Thm 11.3 Let (X_n) be an irreducible recurrent Markov chain with state space S. Let $j \in S$. Define

$$\pi(j) = \frac{1}{\mathbb{E}_j[T_j]} \quad (\pi(j) = 0 \text{ if } \mathbb{E}_j[T_j] = \infty).$$

Let $V_n(j) := \sum_{m=1}^{n} \mathbb{1}_{\{X_m = j\}}$ be the number of visits to state j up to time n. Then for any state $i \in S$

$$\mathbb{P}_i\left[\lim_{n\to\infty} \frac{V_n(j)}{n} = \pi(j)\right] = 1$$

and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{\infty} p_m(i,j) = \pi(j)$$

Proof. (i) $\mathbb{P}_i[\, V_n(j) \to \infty \text{ as } n \to \infty \,] = 1$

Otherwise $\mathbb{P}_j[\,(X_n) \text{ visits } j \text{ finitely many times}\,] > 0$

Ergodic Theorem

Denote by T_j^k the time of the k-th visit to state j.

(ii) $T_j^{V_n(j)} \le n \le T_j^{V_n(j)+1}$

$$V_n(j) = k \Rightarrow T_j^k \le n \quad \text{and} \quad T_j^{k+1} \ge n$$

(iii) $$\mathbb{P}_i\left[\frac{T_j^{V_n(j)}}{V_n(j)} \to \mathbb{E}_j[T_j] \text{ as } n \to \infty \right] = 1$$

Repeating the proof from Thm 10.2 we have that

$$\mathbb{P}_i\left[\frac{T_j^k}{k} \to \mathbb{E}_j[T_j] \text{ as } k \to \infty \right] = 1$$

By (i) $\lim_{n\to\infty} \frac{T_j^{V_n(j)}}{V_n(j)} = \lim_{k\to\infty} \frac{T_j^k}{k}$

(iv) By the Squeeze lemma $\mathbb{P}_i\left[\lim_{n\to\infty} \frac{n}{V_n(j)} = \mathbb{E}_j[T_j]\right] = 1$, therefore

$\mathbb{P}_i\left[\lim_{n\to\infty} \frac{V_n(j)}{n} = \frac{1}{\mathbb{E}_j[T_j]}\right]$. By definition $\pi(j) = \frac{1}{\mathbb{E}_j[T_j]}$.

Ergodic Theorem

(v) $\frac{1}{n}\sum_{m=1}^{n} p_m(i,j) = \frac{1}{n}\mathbb{E}_i\left[\sum_{m=1}^{n} \mathbb{1}_{\{X_m=j\}}\right] = \frac{1}{n}\mathbb{E}_i[V_n(j)]$

(vi) $\lim_{n\to\infty}\frac{1}{n}\mathbb{E}_i[V_n(j)] = \mathbb{E}_i\left[\lim_{n\to\infty}\frac{1}{n}V_n(j)\right] = \pi(j)$ ∎

Convergence theorem

Theorem 7.4 Let P be a transition matrix for a finite-state, irreducible, aperiodic Markov chain. Then there exists a unique stationary distribution π, $\pi = \pi P$, and for any initial probability distribution ν

$$\lim_{n \to \infty} \nu P^n = \pi$$

Theorem 12.1 Let (X_n) be an irreducible aperiodic Markov chain possessing a stationary distribution π. Then for any states i, j

$$\lim_{n \to \infty} P_n(i,j) = \pi(j) \quad \left(= \frac{1}{\mathbb{E}_j[T_j]} \right)$$

Remark (1) Thm 12.1 implies that the stationary distribution of an irreducible aperiodic MC is unique.

(2) In fact any irreducible MC has at most one stationary distribution.

Convergence theorem

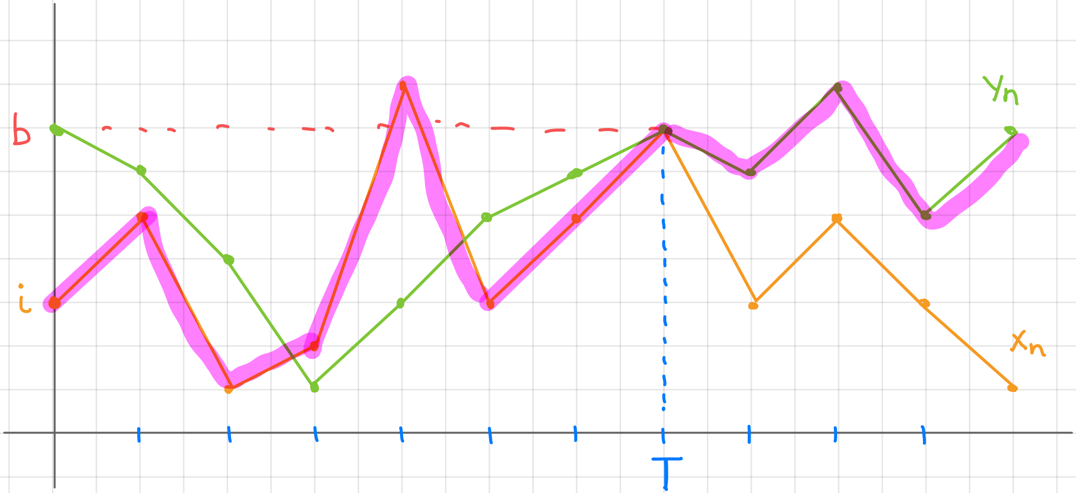
Proof of Thm 12.1

- Idea: couple two independent MCs, one starting from i; another with initial distribution π, wait until they collide.

(X_n) : starting from i

(Y_n) : initial distr. π

$\forall n \quad \mathbb{P}[Y_n = j] = \pi(j)$



(Z_n): starts as (X_n), after T continues as (Y_n)

Convergence theorem

Let $X_0 = i$. Let (Y_n) be a MC with initial distribution π, transition probabilities $p(i,j)$ (same as (X_n)), and independent of (X_n).

Take $b \in S$ and define $T := \{n \geq 1 : X_n = Y_n = b\}$

(i) $\mathbb{P}[T < \infty] = 1$

Consider the process $W_n := (X_n, Y_n)$ on $S \times S$

(W_n) is a MC with $\mathbb{P}[W_0 = (k,\ell)] = \delta_{ki}\,\pi(\ell)$ and transition probabilities $\tilde{p}((k,\ell),(s,t)) = p(k,s)\,p(\ell,t)$

- (X_n) is aperiodic $\Rightarrow \exists\, n$ s.t. $p_n(k,s) > 0$, $p_n(\ell,t) > 0$ $\forall k,s,\ell,t$

  $\Rightarrow \forall\, (k,\ell),(s,t) \in S \times S$ $\tilde{p}_n((k,\ell),(s,t)) = p_n(k,s)\,p_n(\ell,t) > 0$

  $\Rightarrow (W_n)$ is irreducible

Convergence theorem

- $\tilde{\pi}(k,l) := \pi(k)\pi(l)$ is the stationary distribution for (W_n)

$$\sum_{s,t \in S} \tilde{\pi}(s,t)\, p((s,t),(k,l)) = \sum_{s,t \in S} \pi(s)\pi(t)\, p(s,k)\, p(t,l) = \pi(k)\pi(l)$$

- Corollary II.1 $\Rightarrow$ (W_n) is positive recurrent

- $T = \min\{n \geq 1 : W_n = (b,b)\} \overset{HW3,P1}{\Longrightarrow} \mathbb{P}[T < \infty] = 1$

Define $Z_n = \begin{cases} X_n & \text{if } n \leq T \\ Y_n & \text{if } n > T \end{cases}$ $\qquad Z_n' = \begin{cases} Y_n & \text{if } n \leq T \\ X_n & \text{if } n > T \end{cases}$

(ii) (Z_n) is a MC starting from i with transition probabilities $p(i,j)$

- T is the stopping time for (W_n)

Convergence theorem

- By SMP $(X_{T+n}, Y_{T+n})_{n\geq 0}$ is MC starting from (b,b) with transition probabilities $\tilde{p}(i,j)$ independent of $(X_0, Y_0), \ldots, (X_T, Y_T)$
- By symmetry $(Y_{T+n}, X_{T+n})_{n\geq 0}$ is also a MC starting from (b,b) with transition probabilities $\bar{p}(i,j)$ independent of $(X_0, Y_0), \ldots, (X_T, Y_T)$
- Therefore, (Z_n, Z_n') has the same initial distribution and transition probabilities as (X_n, Y_n). In particular, Z_n is a MC starting from i with trans. prob $p(i,j)$

Convergence theorem

(iii) $|\mathbb{P}[X_n = j] - \pi(j)| \leq \mathbb{P}[T \geq n]$

- $\mathbb{P}[Z_n = j] = \mathbb{P}[X_n = j,\ n \leq T] + \mathbb{P}[Y_n = j,\ n > T]$
- By (ii) $\mathbb{P}[Z_n = j] = \mathbb{P}[X_n = j]$
- $\mathbb{P}[Y_n = j] = \pi(j)$
- $|\mathbb{P}[X_n = j] - \pi(j)| = |\mathbb{P}[Z_n = j] - \mathbb{P}[Y_n = j]|$

$$= |\mathbb{P}[X_n = j,\ n \leq T] - \mathbb{P}[Y_n = j,\ n \leq T]| \leq \mathbb{P}[n \leq T]$$

(iv) By (i) $\lim_{n \to \infty} \mathbb{P}[T \geq n] = 0$ ∎