MATH 285: Stochastic Processes

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Today: Continuous time Markov chains

- Homework 5 is due on Sunday, February 20, 11:59 PM

Continuous time Markov chains

<u>Def</u> Let S be a finite or countable state space.

A stochastic process $(X_t)_{t\geq 0}$ with state space S, indexed by non-negative reals t (in the interval $[0,\infty)$, or $[a,b]$) is called a continuous time Markov chain if the following two properties hold:

(1) [Markov property] Let $0 \leq t_0 < t_1 < \cdots < t_n < \infty$ be a sequence of times, and let $i_0, i_1, \ldots, i_n \in S$ be a sequence of states such that $\mathbb{P}[X_{t_0} = i_0, X_{t_1} = i_1, \ldots, X_{t_{n-1}} = i_{n-1}] > 0$. Then

$$\mathbb{P}[X_{t_n} = i_n \mid X_{t_0} = i_0, \ldots, X_{t_{n-1}} = i_{n-1}] = \mathbb{P}[X_{t_n} = i_n \mid X_{t_{n-1}} = i_{n-1}]$$

(2) [Right-continuity] For $t \geq 0$ and $i \in S$, if $X_t = i$ then there is $\varepsilon > 0$ such that $X_s = i$ for all $s \in [t, t+\varepsilon]$

Continuous time Markov chains

Moreover, we say that (X_t) is time homogeneous if

(3) For any $0 \le s < t < \infty$ and states $i, j \in S$

$$\mathbb{P}[X_t = j \mid X_s = i] = \mathbb{P}[X_{t-s} = j \mid X_0 = i]$$

Recall that the evolution of a discrete time MC can be fully described by the one-step transition probabilities

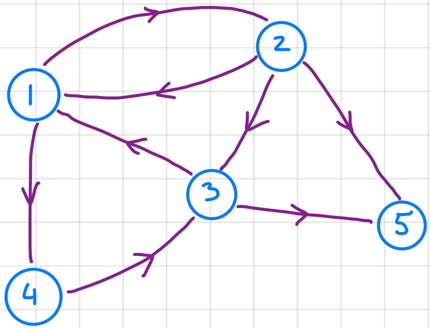
$$\mathbb{P}[X_1 = j \mid X_0 = i] = p(i,j)$$

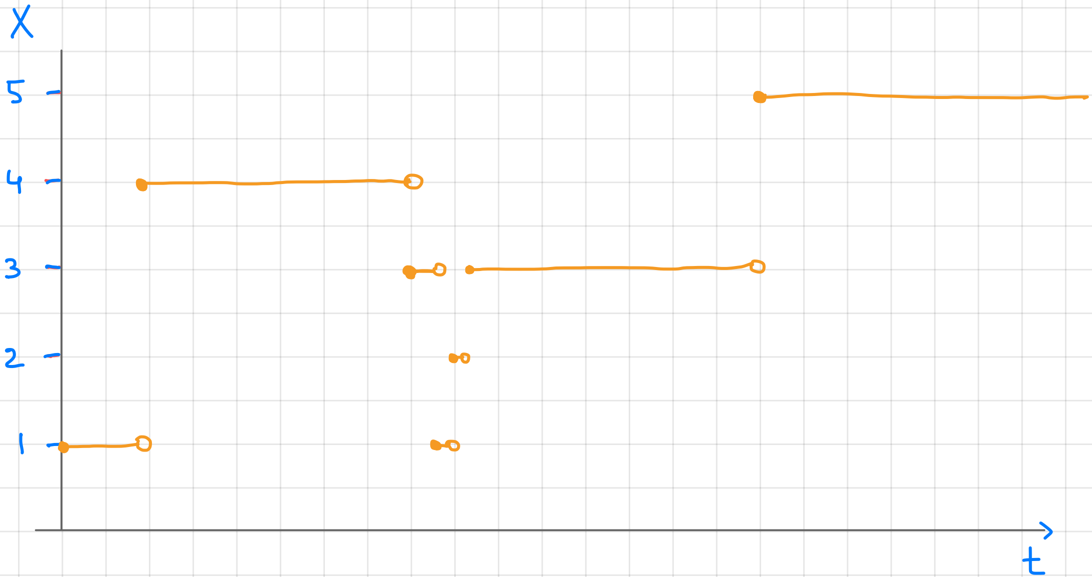
For the continuous time Markov chains we need to know the transition probabilities for infinitely many times

$$p_t(i,j) := \mathbb{P}[X_t = j \mid X_0 = i], \; t > 0 \quad \text{(transition kernel)}$$

(for any fixed i,j $p_t(i,j)$ is a function of t)

Typical trajectory





Jump times

Denote

$$J_1 := \min\{t \geq 0 : X_t \neq X_0\}$$

Right-continuity: if $X_0 = i$ then there exists $\varepsilon > 0$ s.t.

$X_s = i$ for $s \in [0, \varepsilon]$, therefore $\mathbb{P}[J_1 > 0] = 1$

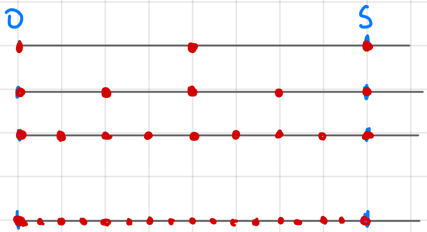
Suppose we have been waiting for a jump for time s, i.e., $J_1 > s$. How much longer are we going to wait?

What is the conditional probability of $J_1 > s+t$ given $J_1 > s$?

Proposition 18.1 For $s, t > 0$ and $i \in S$

$$\mathbb{P}[J_1 > s+t \mid J_1 > s] = \mathbb{P}[J_1 > t]$$

Proof.



$\frac{sj}{2}$, $j \in \{0,1,2\}$

$\frac{sj}{2^2}$, $j \in \{0,1,\dots,2^2\}$

$\frac{sj}{2^k}$, $j \in \{0,1,\dots,2^k\}$

Jump times

Suppose $X_0 = i$.

(1) Denote $A_k = \{ X_{\frac{sj}{2^k}} = i$ for all $j \in \{0, 1, \dots, 2^k\} \}$

Then $\mathbb{P}[J_1 > s] = \mathbb{P}[\bigcap_{k=1}^{\infty} A_k]$

- $\forall k \ \forall j \in \{0, 1, \dots, 2^k\}$ $\frac{sj}{2^k} \in [0, s]$, so $J_1 > s \Rightarrow X_{\frac{sj}{2^k}} = i$
- If $J_1 \leq s$, then $\exists\, s' \in [0, s]$ s.t. $X_{s'} \neq i$. Since X_t is right-continuous, there exists $\varepsilon > 0$ s.t. $\forall u \in [s', s'+\varepsilon]$ $X_u \neq i$. Then there exists k' and j' s.t. $\frac{sj'}{2^{k'}} \in [s', s'+\varepsilon]$, and thus $A_{k'}$ does not hold. So

$$\{J_1 > s\}^c \subset \left\{ \bigcap_{k=1}^{\infty} A_k \right\}^c$$

(2) $\forall k \in \mathbb{N}$ $A_k \supset A_{k+1}$

For all $j \in \{0, 1, \dots, 2^k\}$ $X_{\frac{sj}{2^k}} = X_{\frac{s2j}{2^{k+1}}} = i$ and $2j \in \{0, 1, \dots, 2^{k+1}\}$

Jump times

(3) By the continuity of the probability measure

$$\mathbb{P}[J_1 > s] = \mathbb{P}\Big[\bigcap_{k=1}^{\infty} A_k\Big] = \lim_{k\to\infty} \mathbb{P}[A_k]$$

(4) Denote $B_k = \{ X_{\frac{tj}{2^k}} = i \text{ for all } j \in \{0, 1, \dots, 2^k\} \}$

$C_k = \{ X_{\frac{sj}{2^k}} = i \text{ for all } j \in \{0, 1, \dots, 2^k\}$ and

$X_{s + \frac{tj'}{2^k}} = i \text{ for all } j' \in \{0, 1, \dots, 2^k\} \}$

Then $B_k \supset B_{k+1}$, $C_k \supset C_{k+1}$, and

$$\mathbb{P}[J_1 > t] = \mathbb{P}\Big[\bigcap_{k=1}^{\infty} B_k\Big] = \lim_{k\to\infty} \mathbb{P}[B_k], \quad \mathbb{P}[J_1 > s+t] = \mathbb{P}\Big[\bigcap_{k=1}^{\infty} C_k\Big] = \lim_{k\to\infty} \mathbb{P}[C_k]$$

Jump times

(5) $\mathbb{P}[A_k] = \left(\mathbb{P}[X_{\frac{s}{2^k}} = i \mid X_0 = i]\right)^{2^k}$

$$\mathbb{P}[A_k] = \mathbb{P}[X_0 = i, X_{\frac{s}{2^k}} = i, \dots, X_{\frac{s2^k}{2^k}} = i]$$

$$= \mathbb{P}[X_{\frac{s2^k}{2^k}} = i \mid X_{\frac{s(2^k-1)}{2^k}} = i] \cdot \dots \cdot \mathbb{P}[X_{\frac{s}{2^k}} = i \mid X_0 = i]\,\mathbb{P}[X_0 = i]$$

$$= \left(\mathbb{P}[X_{\frac{s}{2^k}} = i \mid X_0 = i]\right)^{2^k}$$

(6) Similarly $\mathbb{P}[B_k] = \left(\mathbb{P}[X_{\frac{t}{2^k}} = i \mid X_0 = i]\right)^{2^k}$ and

$$\mathbb{P}[C_k] = \left(\mathbb{P}[X_{\frac{s}{2^k}} = i \mid X_0 = i]\right)^{2^k} \left(\mathbb{P}[X_{\frac{t}{2^k}} = i \mid X_0 = i]\right)^{2^k}$$

(7) $\forall k$ $\mathbb{P}[C_k] = \mathbb{P}[A_k]\,\mathbb{P}[B_k] \Rightarrow \lim_{k\to\infty} \mathbb{P}[C_k] = \lim_{k\to\infty} \mathbb{P}[A_k] \lim_{k\to\infty} \mathbb{P}[B_k]$

Finally $\mathbb{P}[J_1 > s+t] = \mathbb{P}[J_1 > s]\,\mathbb{P}[J_1 > t]$ $\blacksquare$

Exponential distribution

$\mathbb{P}[J_1 > s+t \mid J_1 > s] = \mathbb{P}[J_1 > t]$ is called the memoryless property

There is a unique one-parameter family of distributions on $(0,\infty)$ that possesses the memoryless property.

Prop. 18.2 If T is a random variable taking values in $(0,\infty)$ and if T has the memoryless property $\mathbb{P}[T>s+t \mid T>s] = \mathbb{P}[T>t]$ for all $s,t>0$, then T is an exponential random variable with some intensity $q>0$: $\mathbb{P}[T>t] = e^{-qt}$, $t>0$, $\left(f_T(t) = q e^{-qt} \right)$

Proof. Denote $G(t) = \mathbb{P}[T>t]$ and $G(1) = e^{-q}$. Then $G(t+s) = G(t)G(s)$

- $\exists\, n_0$ s.t. $G(\frac{1}{n_0}) > 0 \Rightarrow G(1) = \left(G(\frac{1}{n_0})\right)^{n_0} > 0 \Rightarrow \exists\, q>0$ s.t. $G(1) = e^{-q}$
- $\forall n \in \mathbb{N}$ $G(\frac{1}{n}) = e^{-q\frac{1}{n}}$, $\forall \frac{m}{n} \in \mathbb{Q}_+$ $G\left(\frac{m}{n}\right) = e^{-q\frac{m}{n}}$ $\left(G(t) = e^{-qt} \text{ for } t \in \mathbb{Q}_+ \right)$
- $G(t)$ is decreasing, so if $(t_n), (t_n') \subset \mathbb{Q}_+$, $t_n \nearrow t$, $t_n' \searrow t$

$$e^{-qt} = \lim_{n\to\infty} e^{-qt_n} \le G(t) \le \lim_{n\to\infty} e^{-qt_n'} = e^{-qt} \quad \blacksquare$$

Exponential distribution

We write $T \sim \mathrm{Exp}(q)$. Here are some properties of exponential distribution

Prop. 18.3 Let $T_1, T_2, \dots, T_n$ be independent with $T_j \sim \mathrm{Exp}(q_j)$

(a) Density $f_{T_j}(t) = q_j e^{-q_j t}$, $\mathbb{E}[T_j] = \frac{1}{q_j}$, $\mathrm{Var}[T_j] = \frac{1}{q_j^2}$

(b) $\mathbb{P}[T_j > s+t \mid T_j > s] = \mathbb{P}[T_j > t]$

(c) $T = \min_j T_j$ is exponential with $T \sim \mathrm{Exp}(q_1 + \cdots + q_n)$. Moreover

$$\mathbb{P}[T = T_j] = \frac{q_j}{q_1 + \cdots + q_n}$$