MATH 285: Stochastic Processes

math-old.ucsd.edu/~ynemish/teaching/285

Today: Continuous time Markov chains

- Homework 5 is due on Sunday, February 20, 11:59 PM

Continuous time Markov chains

<u>Def</u> Let S be a finite or countable state space.
A stochastic process $(X_t)_{t\geq 0}$ with state space S, indexed by non-negative reals t (in the interval $[0,\infty)$, or $[a,b]$) is called a continuous time Markov chain if the following two properties hold:

(1) [Markov property] Let $0 \leq t_0 < t_1 < \cdots < t_n < \infty$ be a sequence of times, and let $i_0, i_1, \ldots, i_n \in S$ be a sequence of states such that $\mathbb{P}[X_{t_0} = i_0, X_{t_1} = i_1, \ldots, X_{t_{n-1}} = i_{n-1}] > 0$. Then

(2) [Right-continuity] For $t \geq 0$ and $i \in S$, if then there is $\varepsilon > 0$ such that

Continuous time Markov chains

Moreover, we say that (X_t) is time homogeneous if

(3) For any $0 \leq s < t < \infty$ and states $i, j \in S$

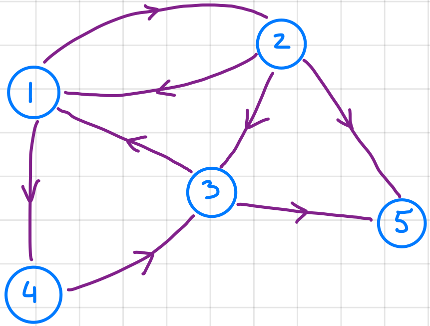
Recall that the evolution of a discrete time MC can be fully described by the one-step transition probabilities

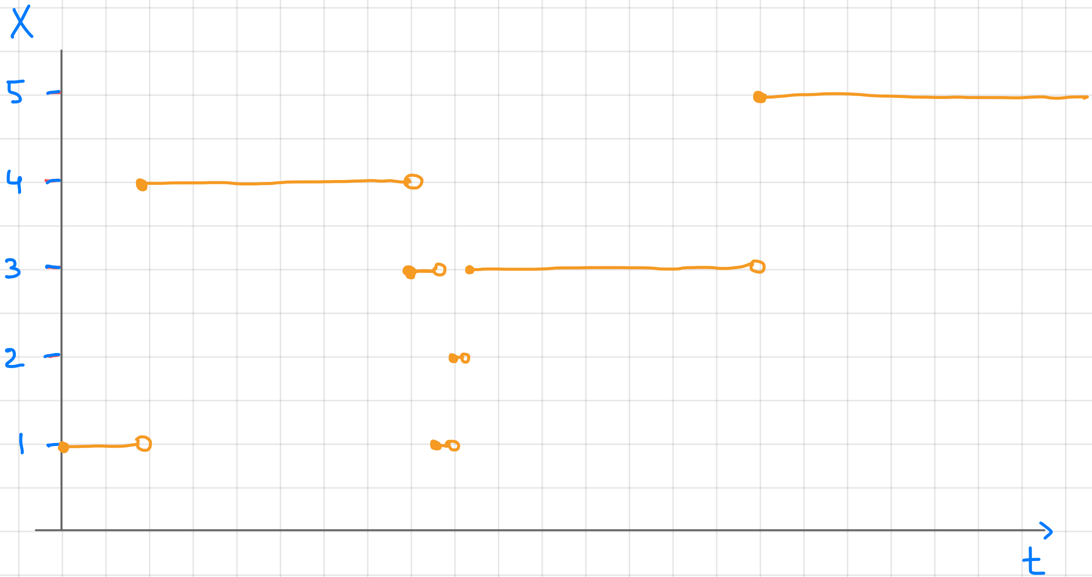
$$\mathbb{P}[X_1 = j \mid X_0 = i] = p(i,j)$$

For the continuous time Markov chains we need to know the transition probabilities for infinitely many times (transition kernel)

(for any fixed i, j $\; p_t(i,j)$ is a function of t)

Typical trajectory





Jump times

Denote

$$J_1 :=$$

Right-continuity: if $X_0 = i$ then there exists $\varepsilon > 0$ s.t.

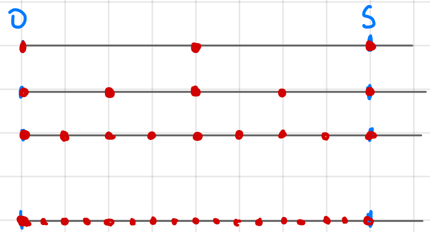
$X_s = i$ for $s \in [0, \varepsilon]$, therefore

Suppose we have been waiting for a jump for time s, i.e., $J_1 > s$. How much longer are we going to wait?

What is the conditional probability of $J_1 > s+t$ given $J_1 > s$?

Proposition 18.1 For $s, t > 0$ and $i \in S$

Proof.



Jump times

Suppose $X_0 = i$.

(1) Denote $A_k =$

Then $\mathbb{P}[J_1 > s] =$

- $\forall k \ \forall j \in \{0, 1, \dots, 2^k\}$, so
- If $J_1 \le s$, then . Since X_t is right-continuous, there exists $\varepsilon > 0$ s.t. . Then there exists k' and j' s.t. , and thus . So

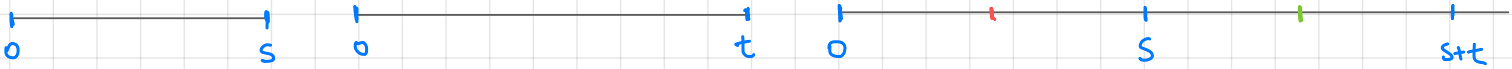
(2) $\forall k \in \mathbb{N}$

For all $j \in \{0, 1, \dots, 2^k\}$ and $2j \in \{0, 1, \dots, 2^{k+1}\}$

Jump times

(3) By the continuity of the probability measure

$$\mathbb{P}[J_1 > s] = \mathbb{P}\Big[\bigcap_{k=1}^{\infty} A_k\Big] =$$



(4) Denote $B_k = \left\{ X_{\frac{tj}{2^k}} = i \text{ for all } j \in \{0, 1, \dots, 2^k\} \right\}$

$C_k = \left\{ X_{\frac{sj}{2^k}} = i \text{ for all } j \in \{0, 1, \dots, 2^k\} \text{ and } X_{s+\frac{tj'}{2^k}} \text{ for all } j' \in \{0, 1, \dots, 2^k\} \right\}$

Then , , and

$$\mathbb{P}[J_1 > t] = \mathbb{P}\Big[\bigcap_{k=1}^{\infty} B_k\Big] = \qquad , \qquad \mathbb{P}[J_1 > s+t] = \mathbb{P}\Big[\bigcap_{k=1}^{\infty} C_k\Big] =$$

Jump times

(5) $\mathbb{P}[A_k] =$

$\mathbb{P}[A_k] = \mathbb{P}[X_0 = i, X_{\frac{s}{2^k}} = i, \dots, X_{\frac{s2^k}{2^k}} = i]$

$=$

$=$

(6) Similarly $\mathbb{P}[B_k] = \left(\mathbb{P}[X_{\frac{t}{2^k}} = i \mid X_0 = i]\right)^{2^k}$ and

$\mathbb{P}[C_k] = \left(\mathbb{P}[X_{\frac{s}{2^k}} = i \mid X_0 = i]\right)^{2^k} \left(\mathbb{P}[X_{\frac{t}{2^k}} = i \mid X_0 = i]\right)^{2^k}$

(7) $\forall k$ $\mathbb{P}[C_k] =$ $\Rightarrow \lim_{k \to \infty} \mathbb{P}[C_k] =$

Finally $\mathbb{P}[J_1 > s+t] =$

Exponential distribution

$\mathbb{P}[J_1 > s+t \mid J_1 > s] = \mathbb{P}[J_1 > t]$ is called the memoryless property

There is a unique one-parameter family of distributions on $(0,\infty)$ that possesses the memoryless property.

Prop. 18.2 If T is a random variable taking values in $(0,\infty)$ and if T has the memoryless property $\mathbb{P}[T > s+t \mid T > s] = \mathbb{P}[T > t]$ for all $s,t>0$, then T is an with some intensity $q>0$:

Proof. Denote $G(t) =$ and . Then $G(t+s) = G(t)G(s)$

- $\exists\, n_0$ s.t. $\Rightarrow G(1) =$ $\Rightarrow \exists\, q>0$ s.t.
- $\forall n \in \mathbb{N}$ $G(\frac{1}{n}) = e^{-q\frac{1}{n}}$, $\forall \frac{m}{n} \in \mathbb{Q}_+$ $G(\frac{m}{n}) =$ ( )
- $G(t)$ is decreasing, so if $(t_n), (t'_n) \subset \mathbb{Q}$, $t_n \nearrow t$, $t'_n \searrow t$

Exponential distribution

We write . Here are some properties of exponential distribution

Prop. 18.3 Let $T_1, T_2, \dots, T_n$ be independent with $T_j \sim Exp(q_j)$

(a) Density $f_{T_j}(t) = q_j e^{-q_j t}$, $\mathbb{E}[T_j] = \frac{1}{q_j}$, $Var[T_j] = \frac{1}{q_j^2}$

(b) $\mathbb{P}[T_j > s+t \mid T_j > s] = \mathbb{P}[T_j > t]$

(c) $T=$ is exponential with $T \sim$. Moreover

$$\mathbb{P}[T = T_j] =$$

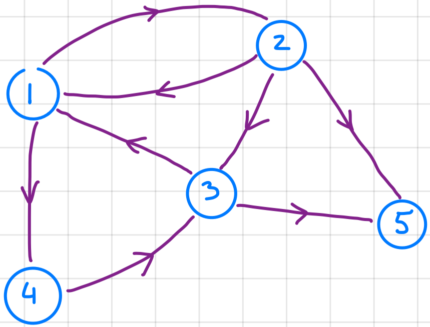
Proof (a), (b) are trivial.

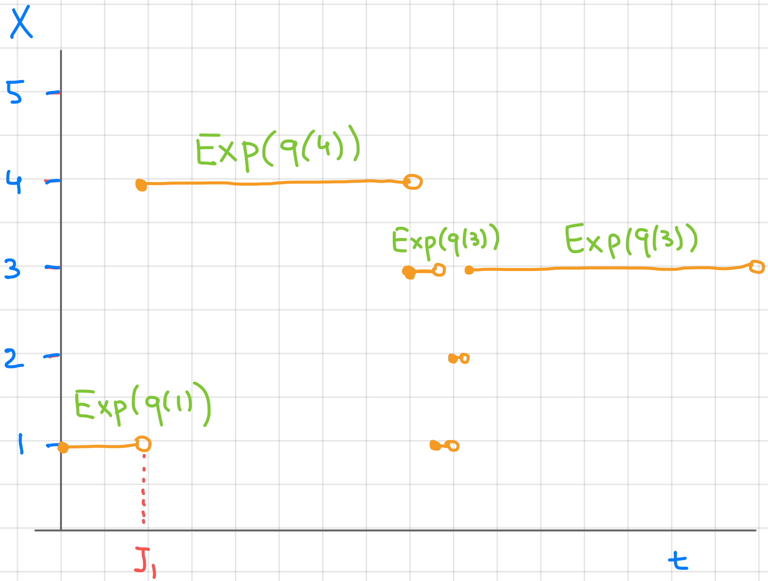
(c) $\mathbb{P}[T > t] = \qquad = \qquad =$

$$\mathbb{P}[T = T_1] = \mathbb{P}[T_2 > T_1, \dots, T_n > T_1] =$$

$$=$$

Transition rates





- Conditioned on $X_0 = i$,

- Denote $p(i,j) =$

- Define transition rates

$$q(i,j) =$$

$$q(i,j) \geq 0 \, , \quad q(i,i) = 0$$

- $\sum_j q(i,j) =$

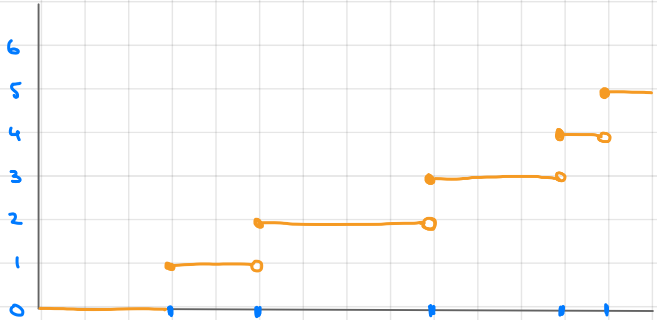
- $p(i,j) =$

Poisson process

Consider a continuous-time MC on the state space $S = \{0, 1, 2, \dots\}$ and transition rates

$$q(i, i+1) = \quad , \qquad q(i,j) = \quad \text{for } j \neq i+1$$

We call this process the Poisson process with rate $\lambda > 0$.



Start a clock $\text{Exp}(\lambda)$.

When it rings, move up.

Repeat ...

Proposition 18.5 Let $(X_t)_{t \geq 0}$ be a Poisson process with rate λ.

The for any $t > 0$, conditioned on $X_0 = 0$,

$$\mathbb{P}[X_t = k] =$$