MATH 285: Stochastic Processes

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Today: Strong Markov property
Embedded jump chain
Infinitesimal description

- Homework 5 is due on Sunday, February 20, 11:59 PM

Exponential distribution

We write $T \sim \text{Exp}(q)$. Here are some properties of exponential distribution

Prop. 18.3 Let $T_1, T_2, \dots, T_n$ be independent with $T_j \sim \text{Exp}(q_j)$

(a) Density $f_{T_j}(t) = q_j e^{-q_j t}$, $\mathbb{E}[T_j] = \frac{1}{q_j}$, $\text{Var}[T_j] = \frac{1}{q_j^2}$

(b) $\mathbb{P}[T_j > s+t \mid T_j > s] = \mathbb{P}[T_j > t]$

(c) $T = \min_j T_j$ is exponential with $T \sim \text{Exp}(q_1 + \dots + q_n)$. Moreover

$$\mathbb{P}[T = T_j] = \frac{q_j}{q_1 + \dots + q_n}$$

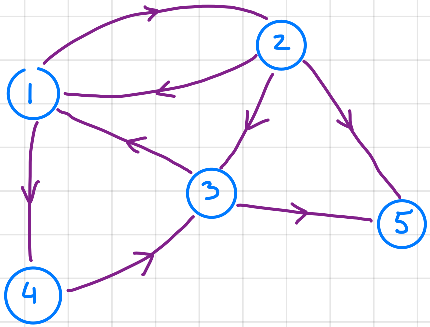
Proof. (a), (b) are trivial.

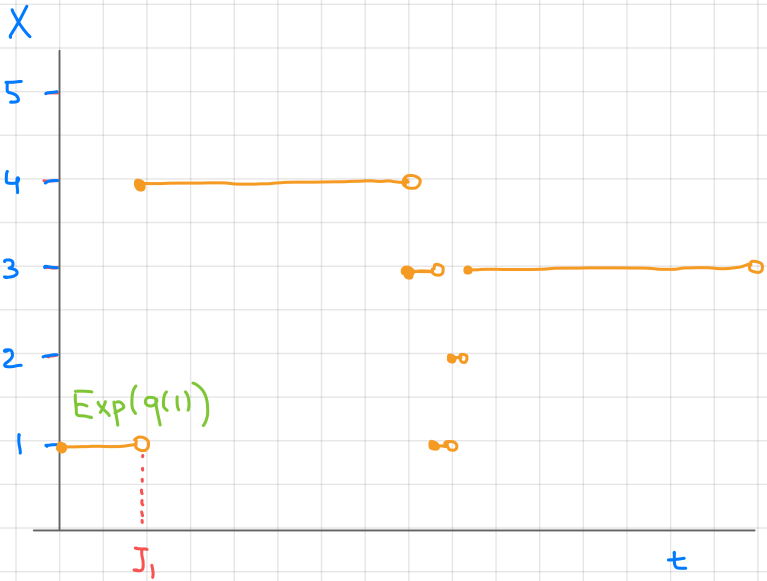
(c) $\mathbb{P}[T > t] = \qquad = \qquad =$

$\mathbb{P}[T = T_1] = \mathbb{P}[T_2 > T_1, \dots, T_n > T_1] =$

$=$

Transition rates





- Conditioned on $X_0 = i$,

- Denote $p(i,j) =$

  $p(i,i) =$

- Define transition rates

  $q(i,j) =$

  $q(i,j) \geq 0$, $q(i,i) = 0$

- $\sum_j q(i,j) =$

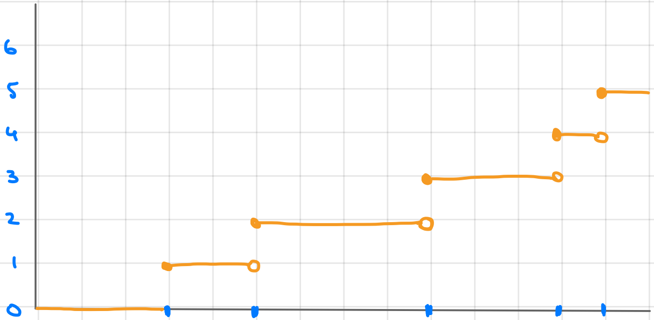
- $p(i,j) =$

Poisson process

Consider a continuous-time MC on the state space $S = \{0, 1, 2, \ldots\}$ and transition rates

$$q(i, i+1) = \quad , \qquad q(i,j) = \quad \text{for } j \neq i+1$$

We call this process the Poisson process with rate $\lambda > 0$.



Start a clock $\mathrm{Exp}(\lambda)$.

When it rings, move up.

Repeat ...

Proposition 18.5 Let $(X_t)_{t \geq 0}$ be a Poisson process with rate λ.

The for any $t > 0$, conditioned on $X_0 = 0$,

$$\mathbb{P}[X_t = k] =$$

Strong Markov property

Given a MC $(X_t)_{t\geq 0}$, a stopping time T is a random variable taking values in $[0, +\infty]$ with property that

Thm 19.1 (Strong Markov property) Let $(X_t)_{t\geq 0}$ be a continuous-time MC with state space S and transition rates $q(i,j)$, $i,j \in S$. Let T be a stopping time. For some $i>0$, suppose that $\mathbb{P}[X_T = i] > 0$. Then, is

a MC with the same

No proof. Strong Markov property can be used to develop the first step analysis.

First step analysis

For any set $A \subset S$ denote the hitting time

$$\tau_A =$$

- For $A, B \subset S$, $A \cap B = \emptyset$, what is the probability of reaching A before B?

Set $h(i) = \mathbb{P}_i[\tau_A < \tau_B]$. Then

$(*)$ $\left\{ \right.$

$$h(i) = \sum_{j \in S} \mathbb{P}_i[X_{J_1} = j]\, \mathbb{P}_i[\tau_A < \tau_B \mid X_{J_1} = j] =$$

$$\mathbb{P}_i[\tau_A < \tau_B \mid X_{J_1} = j] \overset{SMP}{=}$$

First step analysis

- Expected hitting time: $\mathbb{E}_i[\tau_A]$

Denote $g(i) := \mathbb{E}_i[\tau_A]$. Then

$$(**) \quad \begin{cases} \\ \\ \end{cases}$$

$$g(i) = \sum_{j \in S} \mathbb{P}_i[X_{J_1} = j]\, \mathbb{E}_i[\tau_A \mid X_{J_1} = j] =$$

Define $Y_t =$. Then

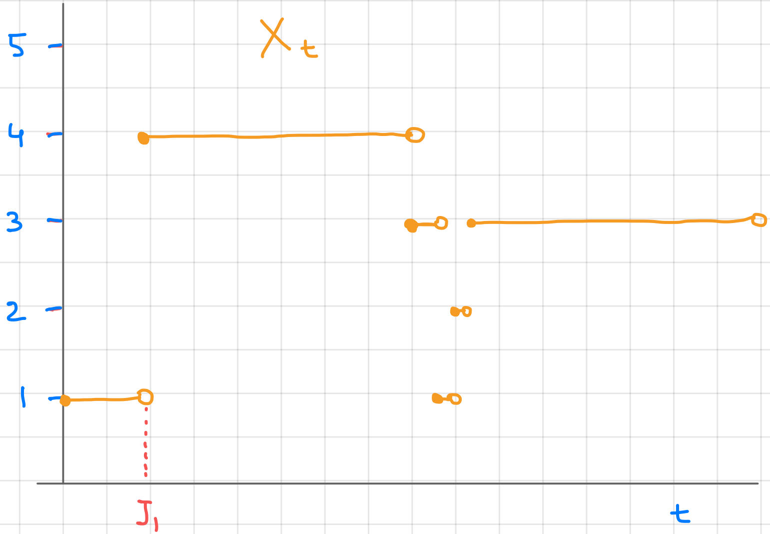
$$\tau_A = \min\{t \geq 0 : X_t \in A\} =$$

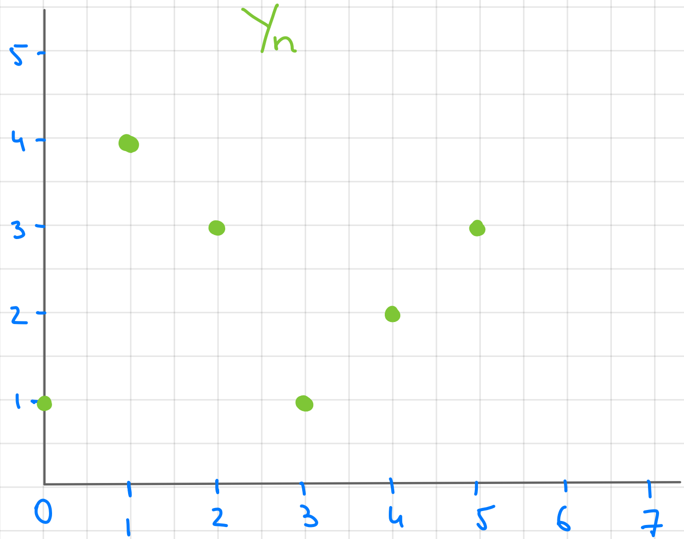
$$=$$

$$\mathbb{E}_i[\tau_A \mid X_{J_1} = j] =$$

$$\Rightarrow g(i) = \sum_{j \in S} \frac{q(i,j)}{q(i)} \left[\frac{1}{q(i)} + g(j) \right] \Rightarrow$$

Embedded jump chain





Denote $J_0 = 0$,

By the strong Markov property, for any $i_0, \dots, i_n \in S$

$$\mathbb{P}[X_{J_n} = i_n, \dots, X_{J_1} = i_1, X_0 = i_0] = \mathbb{P}[X_{J_n} = i_n \mid X_{J_{n-1}} = i_{n-1}, \dots, X_0 = i_0]\, \mathbb{P}[X_{J_{n-1}} = i_{n-1}, \dots, X_0 = i_0]$$

$=$

Denote $Y_n :=$, the embedded jump chain of $(X_t)_{t \geq 0}$.

Embedded jump chain

The embedded jump chain $(Y_n)_{n\geq 0}$ is a discrete-time MC with state space S and transition probabilities

$$\mathbb{P}[Y_1 = j \mid Y_0 = i] = \mathbb{P}[X_{J_1} = j \mid X_0 = i] =$$

What is the distribution of the time between two consecutive jumps? Denote by $S_k :=$ the sojourn times.

We know that $S_1 = J_1 \sim$. Denote $\tilde{X}_t :=$. Given $Y_{k-1} = i_{k-1}$ (and $J_{k-1} < \infty$) by the SMP for (X_t) and J_{k-1}, the first jump time of $\tilde{X}_t$ has exponential distribution $\tilde{J}_1 =$

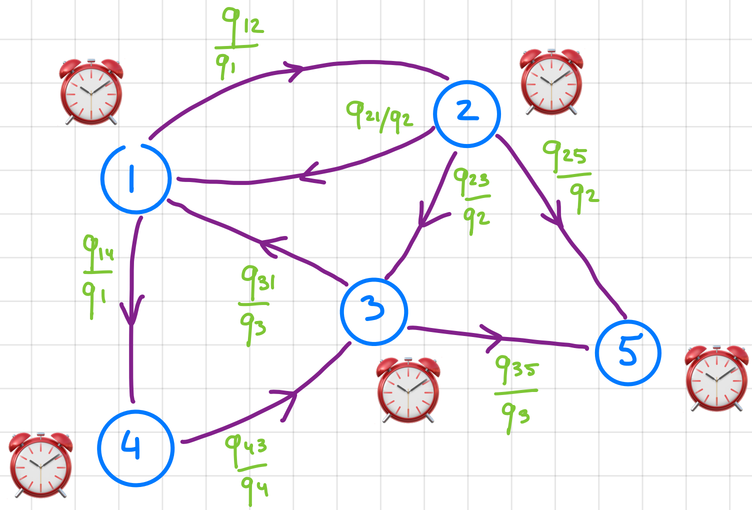
$\mathbb{P}[\tilde{X}_{\tilde{J}_1} = i_k] = \mathbb{P}[Y_k = i_k] =$, S_k, Y_k are indep. and indep. of $S_1, \dots, S_{k-1}$, $Y_0, \dots, Y_{k-1}$

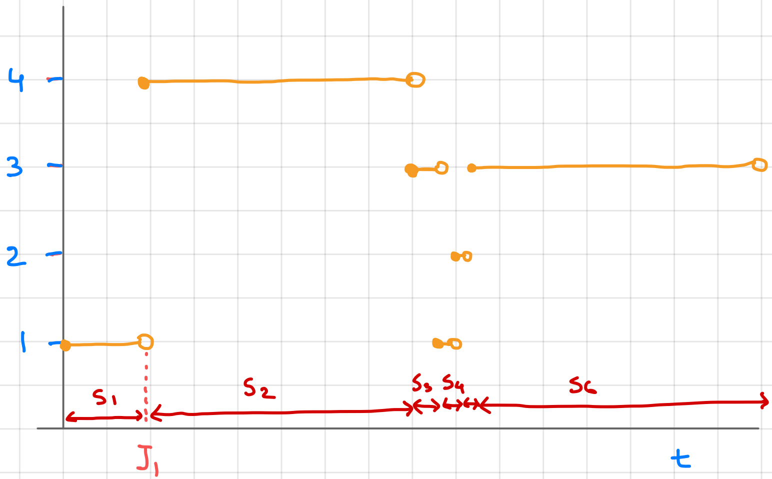
Prop. 19.2 Conditioned on $Y_0, \dots, Y_{n-1}$, the sojourn times $S_1, \dots, S_n$ are independent exponential random variables with

Embedded jump chain

Jump and hold construction

- embedded jump chain (Y_n) with $\mathbb{P}[Y_{n+1}=j \mid Y_n=i] = \frac{q_{ij}}{q_i}$
- exponential sojourn times S_n with $S_n \sim \text{Exp}(q(Y_{n-1}))$





- start from $X_0 = Y_0 = i_0$
- wait at i_0 $S_1 \sim \text{Exp}(q(i_0))$
- $J_1 = S_1$, $X_{J_1} = Y_1 = i_1$
- wait at i_1 $S_2 \sim \text{Exp}(q(i_1))$
- $J_2 = S_1 + S_2$, $X_{J_2} = Y_2 = i_2$

$\vdots$

Infinitesimal description

Transition rates completely determine the Markov chain.

Q: What is the distribution of X_t ? $\mathbb{P}_i[X_t = j] = p_t(i,j) = ?$

<u>Thm 19.3</u> Let $(X_t)_{t\geq 0}$ be a MC with state space S and transition rates $q(i,j)$. Then the transition probabilities satisfy

$$p_t(i,i) =$$

$$p_t(i,j) =$$

<u>Proof</u>.

(1) $p_t(i,i) = \mathbb{P}_i[X_t = i]$

(2) $p_t(i,j)$

$p_t(i,j) = \mathbb{P}_i[X_t = j] \geq$

$=$

$=$

Infinitesimal description

(3) We can write (1) and (2) as

$$p_t(i,i) \geq 1 - q(i)t + \xi_{ii}(t), \quad \xi_{ii}(t) = o(t)$$

$$p_t(i,j) \geq q(i,j)t + \xi_{ij}(t), \quad \xi_{ij}(t) = o(t)$$

Then

$$p_t(i,i) = 1 - q(i)t + \xi_{ii}(t)$$

$$p_t(i,j) = q(i,j)t + \xi_{ij}(t)$$

Take the sum

$$p_t(i,i) + \sum_{j \neq i} p_t(i,j) =$$

$\Rightarrow$ $\Rightarrow$ $\Rightarrow$

<u>Remark</u> In order to identify a Markov chain it is enough to compute $p_t(i,j)$ to first order in t as $t \to 0$.