MATH 285: Stochastic Processes

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Today: Introduction.
Definition of Markov processes

- Test Homework on Gradescope

Stochastic Processes

Def. 1.1 Let T and S be two sets, and let $(\Omega, \mathbb{P})$ be a probability space. We call a collection $(X_t)_{t\in T}$ of random variables that are all defined on the same probability space $(\Omega, \mathbb{P})$ and take values in S a stochastic process indexed by T and taking values in S. If $T = [0, +\infty)$, then $(X_t)_{t\geq 0}$ is called a stochastic process. If $T = \mathbb{N}$, then $(X_n)_{n\in\mathbb{N}}$ is a stochastic process.

T: index set (time), S: state space

Stochastic Processes

Motivation: Mathematical model of phenomena that

Stochastic processes have applications in many disciplines such as biology,[6] chemistry,[7] ecology,[8] neuroscience,[9] physics,[10] image processing, signal processing,[11] control theory,[12] information theory,[13] computer science,[14] cryptography[15] and telecommunications.[16]

en.m.wikipedia.org

+ finance

Prices, sizes of populations, number of particles, ...

Examples

Example 1.2 $X_1, X_2, \dots$ are i.i.d. random variables (real-valued) defined on the same probability space.

Then $(X_n)_{n \in \mathbb{N}}$ is a discrete-time stochastic proeess.

Define

Example 1.3 As above, but

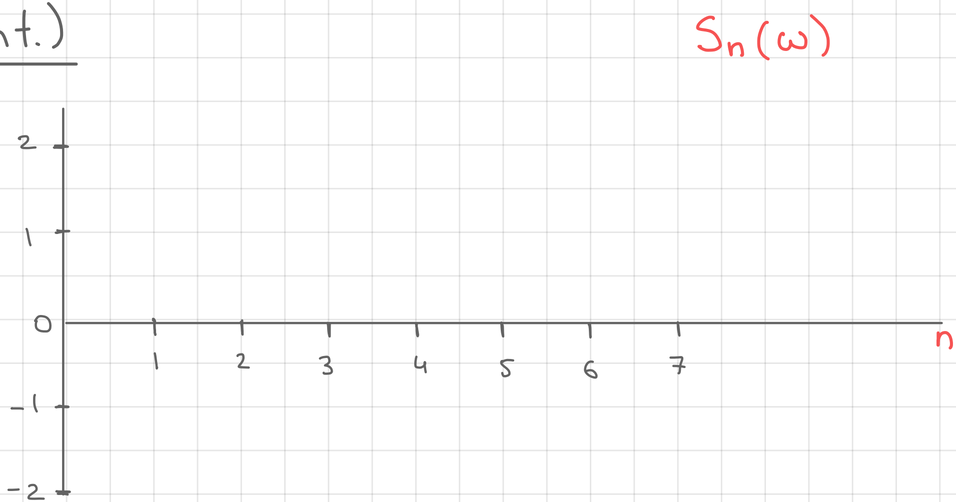
(X_n) :

(S_n) :

Examples

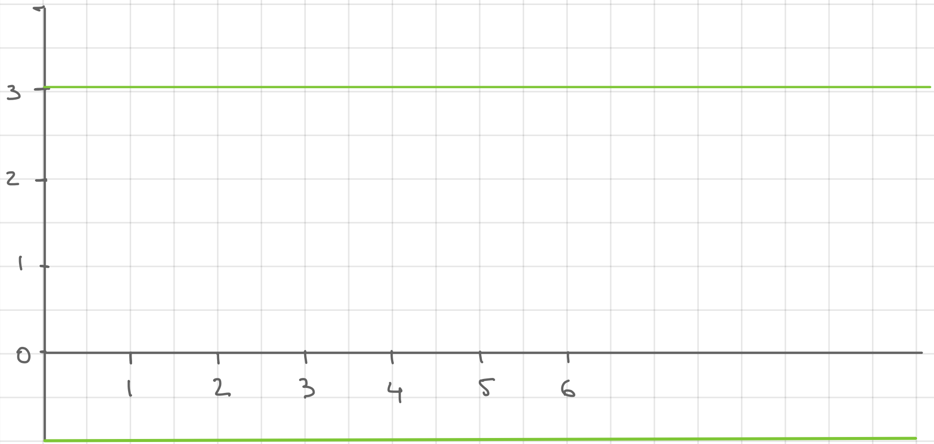
Example 1.3 (cont.)

Random walk



Example 1.4

- reflected RW
- absorbing RW
- partially reflected RW



Discrete time Markov chain

Suppose that S is a discrete state space, and $(X_1, \dots, X_n)$ is a collection of r.v.s with values in S.

Q:

$\mathbb{P}[X_1 = i_1, X_2 = i_2, \dots, X_n = i_n]$

$=$

$=$

$\vdots$

$=$

$=$

Discrete time Markov chain

Def 1.5 Let X_n be a discrete time stochastic process with state space S that is finite or countably infinite. Then X_n is called a if

for each $n \in \mathbb{N}$ and each $(i_1, \dots, i_n) \in S^n$

(M)

Example 1.2 (Recall $\{X_i\}$ are i.i.d.)

Suppose that S is finite or countably infinite

Then (by independence) $\mathbb{P}[X_n = i_n \mid X_1 = i_1, \dots, X_{n-1} = i_{n-1}]$

and $P[X_n = i_n \mid X_{n-1} = i_{n-1}] =$, so (M) is satisfied.

$P[X_1 = i_1, \dots, X_n = i_n] =$

Discrete time Markov chain

<u>Exemple 1.2 (cont.)</u> Recall $S_n = X_1 + \cdots + X_n$, so $X_n =$

and thus $\mathbb{P}[S_1 = i_1, \ldots, S_n = i_n] =$

Check (M)

$\mathbb{P}[S_n = i_n \mid S_1 = i_1, \ldots, S_{n-1} = i_{n-1}] =$

$=$

$=$

$\mathbb{P}[S_n = i_n \mid S_{n-1} = i_n] =$

$=$

We conclude that S_n is

Transition probabilities. Time-homogeneous MC

"Distribution" of a Markov chain is completely described

by the collection

Def. 1.6 A Markov chain is called time-homogeneous

if for any $i, j \in S$

i.e., there exists a function $p : S \times S \to [0,1]$ s.t.

We call $\mathbb{P}[X_n = j \mid X_{n-1} = i]$ the

"Distribution" of a time-homogeneous MC is

determined by the

and

Transition probabilities

If $p(i,j)$ are the transition probabilities, then

$$\sum_{j\in S} p(i,j) =$$

Def. If A is an $n\times n$ matrix s.t. $\forall\, i\in\{1,\dots,n\}$

$$\sum_{j=1}^{n} A_{ij} = 1,$$ then A is called

Suppose $|S|<\infty$ and let

$$P := \left(p(i,j)\right)_{i,j\in S}, \quad P = \begin{array}{c c} & \begin{array}{cccc} s_1 & s_2 & s_3 & \cdots \end{array} \\ \begin{array}{c} s_1 \\ s_2 \\ \vdots \end{array} & \left[\begin{array}{cccc} & & & \\ & & & \\ & & & \end{array}\right] \end{array}$$

Then

Transition probabilities

Example 1.7

Markov chain on $S = \{0, 1, 2, \dots, N\}$

Transition probabilities:

if $i \in \{1, 2, \dots, N-1\}$ then $p(i,j) = \left\{ \right.$

Reflecting random walk:

Absorbing random walk:

Partially reflecting walk: