MATH 285: Stochastic Processes

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Today: Birth and death chains
Recurrence and transience
Stationary distribution

- Homework 6 is due on Friday, March 4, 11:59 PM

Recurrence and transience

Def 21.2 Let $(X_t)_{t\geq 0}$ be a continuous-time MC with state space S, and let $i \in S$. Let $T_i = \min\{t \geq J_1 : X_t = i\}$.

The state i is called transient if $\mathbb{P}_i[T_i < \infty] < 1$

recurrent if $\mathbb{P}_i[T_i < \infty] = 1$

positive recurrent if $\mathbb{E}_i[T_i] < \infty$

- i is recurrent (transient) for (X_t) iff i is recurrent (transient) for the embedded jump chain (Y_n)

  X_t revisits i infinitely many times

  iff Y_n revisits i infinitely many times

  $$T_i = \min\{J_k : k \geq 1, X_{J_k} = i\}$$

- Positive recurrence takes into account how long it takes to revisit i

Recurrence for birth and death chains

Let $(X_t)_{t\geq 0}$ be a birth and death chain with parameters

$\lambda_i = q(i,i+1) > 0$ for $i \geq 0$, $\mu_i = q(i,i-1) > 0$. for $i \geq 1$

$$\begin{tikzcd}
0 \arrow[r, bend left, "\lambda_0"] & 1 \arrow[l, bend left, "\mu_1"] \arrow[r, bend left, "\lambda_1"] & 2 \arrow[l, bend left, "\mu_2"] \arrow[r, bend left, "\lambda_2"] & 3 \arrow[l, bend left, "\mu_3"] \arrow[r, bend left, "\lambda_3"] & 4 \arrow[l, bend left, "\mu_4"] \arrow[r, bend left, "\lambda_4"] & 5 \arrow[l, bend left, "\mu_5"] & \cdots
\end{tikzcd}$$

(X_t) is irreducible (all $\lambda_i > 0$, $\mu_i > 0$), so it is enough to analyze one state for recurrence/transience (take state 0).

Similarly as for the discrete-time MC, denote

$$h(i) := \mathbb{P}_i[\exists\, t \geq 0 : X_t = 0]$$

Then

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$$\begin{cases} h(0) = 1 \\ h(i) = \sum_{j\geq 0} \mathbb{P}_i[\exists\, t\geq 0 : X_t = 0 \mid X_{J_1} = j]\, \mathbb{P}_i[X_{J_1} = j], \quad i > 0 \end{cases}$$

Recurrence for birth and death chains

By the Strong Markov property

$$h(i) = \sum_{j \geq 0} p(i,j)\, h(j) \quad , i \geq 1 \qquad (*)$$

Recall that $p(i,j) = q(i,j)/q(i)$, so $(*)$ becomes

$$h(i) = \sum_{j \geq 0} \frac{q(i,j)}{q(i)} h(j) = \frac{\lambda_i}{\lambda_i + \mu_i} h(i+1) + \frac{\mu_i}{\lambda_i + \mu_i} h(i-1)$$

We can rewrite this using the differences

$$h(i+1) - h(i) = \frac{\mu_i}{\lambda_i} [h(i) - h(i-1)]$$

Applying the above identities recursively gives

$$h(i+1) - h(i) = \underbrace{\frac{\mu_i \mu_{i-1} \cdots \mu_1}{\lambda_i \lambda_{i-1} \cdots \lambda_1}} [h(1) - h(0)]$$

$$= \rho_i [h(1) - h(0)] \qquad i \geq 1$$

Recurrence for birth and death chains

After taking the partial sums

$$h(n) - h(0) = \sum_{i=0}^{n-1} [h(i+1) - h(i)] = [h(1) - h(0)] \sum_{i=0}^{n-1} p_i$$

- if $\sum_{i=0}^{\infty} p_i = \infty$, then $h(1) - h(0) = 0$, and $\forall n \geq 1$ $h(n) = h(0) = 1$

  ↳ (X_t) is recurrent

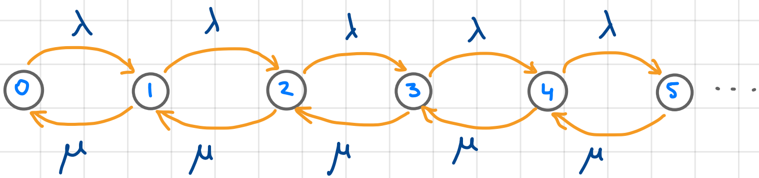
- if $\sum_{i=0}^{\infty} p_i < \infty$, we need to find the minimal solution (Thm 7.0)

  which is achieved when $h(0) - h(1) = \dfrac{1}{\sum_{i=0}^{\infty} p_i}$

Then $h(1) = 1 - \dfrac{1}{\sum_{i=0}^{\infty} p_i} < 1$ and (X_t) is transient.

Example: M/M/1 queueing system

Consider the birth and death chain with $\lambda_i = \lambda$ and $\mu_i = \mu$ (constant and non-zero). Model for a system where jobs (customers) arrive at Poissonian times (at rate λ), queue up, and are executed (served) in the order they arrived at rate μ.



The process (X_t) is the number of jobs (customers) in the queue at time t. From the previous example

$$p_i = \frac{\mu^i}{\lambda^i} = \left(\frac{\mu}{\lambda}\right)^i \text{, and thus } \sum_{i=0}^{\infty} \left(\frac{\mu}{\lambda}\right)^i = \infty \text{ if } \mu \geq \lambda \text{, } \sum_{i=0}^{\infty} \left(\frac{\mu}{\lambda}\right)^i < \infty \text{ if } \mu < \lambda$$

Stationary distribution

Def 22.1 Let (X_t) be a continuous-time MC with transition rates $q(i,j)$. A probability distribution π is called stationary (or invariant) if for each state j

$$q(j)\pi(j) = \sum_i \pi(i)\, q(i,j)$$

Or in terms of the infinitesimal generator

$$\pi A = 0$$

Remark If (Y_n) is the corresponding embedded jump chain, then the stationary distributions for (X_t) and (Y_n) are not the same (and do not necessarily exist simultaneously).

Set $\tilde{\pi}(i) = q(i)\pi(i)$. Then $\tilde{\pi}(j) = q(j)\pi(j) = \sum_i \pi(i)\, q(i) \frac{q(i,j)}{q(i)} = \sum_i \tilde{\pi}(i)\, p(i,j)$

It may be that $\sum \pi(i) < \infty$, but $\sum \tilde{\pi}(i) = \infty$.

Stationary distribution Thm 22.3

Let $(X_t)_{t\geq 0}$ be a continuous time non-explosive MC, and suppose that π is a stationary distribution for (X_t). If $\mathbb{P}[X_0=j]=\pi(j)$ for all states j, then $\mathbb{P}[X_t=j]=\pi(j)\ \forall t>0$

Proof (for finite state space) Fix state j. $\frac{d}{dt}P_t = AP_t$

$$\frac{d}{dt}\mathbb{P}[X_t=j]=\frac{d}{dt}\sum_i \mathbb{P}[X_0=i]\,\mathbb{P}[X_t=j\mid X_0=i]=\frac{d}{dt}\sum_i \pi(i)p_t(i,j)=\sum_i \pi(i)\frac{d}{dt}p_t(i,j)$$

By Kolmogorov's backward equation

$$\frac{d}{dt}p_t(i,j)=\sum_{k\neq i}q(i,k)p_t(k,j)-q(i)p_t(i,j)$$

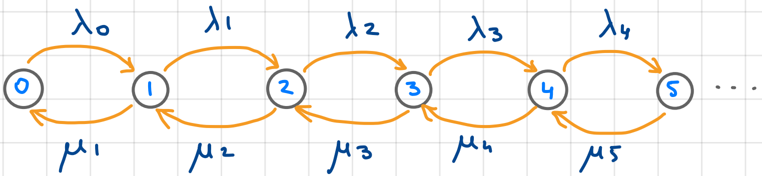
$$\sum_i \pi(i)\frac{d}{dt}p_t(i,j)=\sum_i \pi(i)\sum_k q(i,k)p_t(k,j)-\sum_i \pi(i)q(i)p_t(i,j)$$

$$=\sum_k\sum_i \pi(i)q(i,k)p_t(k,j)-\sum_i q(i)p_t(i,j)\pi(i)$$

$$=\sum_k q(k)\pi(k)p_t(k,j)-\sum_i q(i)\pi(i)p_t(i,j)=0 \quad \blacksquare$$

Example: Irreducible birth and death chain

Let (X_t) be an irreducible birth and death chain.



$\lambda_i > 0$

$\mu_i > 0$

Equations: $\lambda_0 \pi(0) = \mu_1 \pi(1)$, $(\lambda_j + \mu_j)\pi(j) = \mu_{j+1}\pi(j+1) + \lambda_{j-1}\pi(j-1)$

Rewrite
$$\begin{cases} \lambda_0 \pi(0) = \mu_1 \pi(1) \\ \mu_{j+1}\pi(j+1) - \mu_j \pi(j) = \lambda_j \pi(j) - \lambda_{j-1}\pi(j-1) \end{cases}$$

$j=1$: $\mu_2 \pi(2) - \mu_1 \pi(1) = \lambda_1 \pi(1) - \lambda_0 \pi(0) = \lambda_1 \pi(1) - \mu_1 \pi(1)$

$\mu_2 \pi(2) = \lambda_1 \pi(1)$

$\vdots$

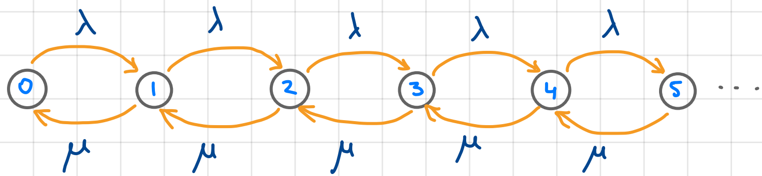
$\mu_{j+1}\pi(j+1) = \lambda_j \pi(j)$

$$\Rightarrow \pi(j+1) = \frac{\lambda_j}{\mu_{j+1}}\pi(j) = \overbrace{\frac{\lambda_j \cdots \lambda_0}{\mu_{j+1} \cdots \mu_1}}^{\theta_{j+1}} \pi(0)$$

Set $\theta_0 = 1$. Then $\pi(j) = \theta_j \pi(0)$ for all j, and the stationary distribution exists iff $\sum_{j=0}^{\infty} \theta_j < \infty$, in which case $\pi(0) = \frac{1}{\sum_{j=0}^{\infty}\theta_j}$

Example: M/M/1 queue

Let (X_t) be an M/M/1 queue



From the previous example

$$\theta_j = \frac{\lambda_{j-1}\cdots\lambda_0}{\mu_j\cdots\mu_1} = \left(\frac{\lambda}{\mu}\right)^j$$

The stationary distribution exists iff $\sum_{j=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^j < \infty$

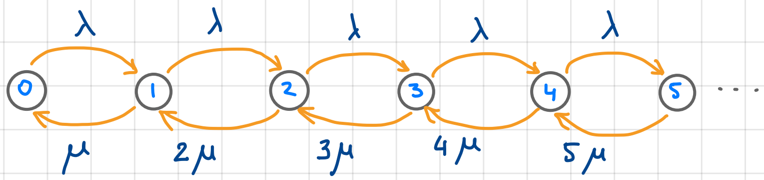
$\sum_{j=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^j < \infty$ iff $\lambda < \mu$ in which case $\sum_{j=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^j = \frac{1}{1-\frac{\lambda}{\mu}}$

and $\pi(0) = 1 - \frac{\lambda}{\mu}$

$$\pi(j) = \theta_j \pi_0 = \left(\frac{\lambda}{\mu}\right)^j\left(1 - \frac{\lambda}{\mu}\right) \leftarrow \text{sGeom}\left(\frac{\lambda}{\mu}\right)$$

Example: M/M/∞ queue

Queue with infinitely many servers



$\lambda_i = \lambda$

$\mu_i = i\mu$

Repeating the same argument as in the previous example

$$\theta_j = \frac{\lambda_{j-1} \cdots \lambda_0}{\mu_j \cdots \mu_1} = \left(\frac{\lambda}{\mu}\right)^j \cdot \frac{1}{j!}$$

$\sum_{i=0}^{\infty} \theta_i = \sum_{j \geq 0} \left(\frac{\lambda}{\mu}\right)^j \frac{1}{j!} = e^{\frac{\lambda}{\mu}} < \infty$ for all $\lambda > 0, \mu > 0$, so the stationary distribution always exists

$$\pi(0) = e^{-\frac{\lambda}{\mu}}, \quad \pi(j) = \frac{1}{j!}\left(\frac{\lambda}{\mu}\right)^j e^{-\frac{\lambda}{\mu}} \leftarrow \mathrm{Pois}\left(\frac{\lambda}{\mu}\right)$$