MATH 285: Stochastic Processes

math-old.ucsd.edu/~ynemish/teaching/285

Today: Birth and death chains
Recurrence and transience
Stationary distribution

- Homework 6 is due on Friday, March 4, 11:59 PM

Recurrence and transience

Def 21.2 Let $(X_t)_{t\geq 0}$ be a continuous-time MC with state space S, and let $i\in S$. Let $T_i = \min\{t>0 : X_t = i\}$.

The state i is called transient if $\mathbb{P}_i[T_i<\infty] = 0$.

recurrent if $\mathbb{P}_i[T_i<\infty] = 1$

positive recurrent if $\mathbb{E}_i[T_i] < \infty$

- i is recurrent (transient) for (X_t) iff i is recurrent (transient) for the embedded jump chain (Y_n)

  X_t revisits i infinitely many times

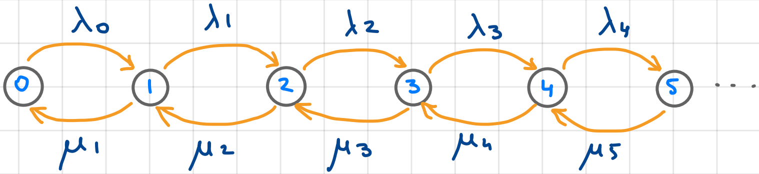
  iff Y_n revisits i infinitely many times

- Positive recurrence takes into account how long it takes to revisit i

Recurrence for birth and death chains

Let $(X_t)_{t\geq 0}$ be a birth and death chain with parameters

$\lambda_i = q(i,i+1) > 0$ for $i \geq 0$, $\mu_i = q(i,i-1) > 0$. for $i \geq 1$



(X_t) is irreducible (all $\lambda_i > 0$, $\mu_i > 0$), so it is enough to analyze one state for recurrence/transience (take state 0).

Similarly as for the discrete-time MC, denote

$h(i) :=$

Then

FSA

Recurrence for birth and death chains

By the Strong Markov property

$$h(i) = \qquad (*)$$

Recall that $p(i,j) = q(i,j)/q(i)$, so $(*)$ becomes

$$h(i) =$$

We can rewrite this using the differences

$$h(i+1) - h(i) =$$

Applying the above identities recursively gives

$$h(i+1) - h(i) =$$

$$=$$

Recurrence for birth and death chains

After taking the partial sums

$$h(n) - h(0) =$$

- if $\sum_{i=0}^{\infty} \rho_i = \infty$, then , and $\forall n \geq 1$

  ↳ (X_t) is recurrent

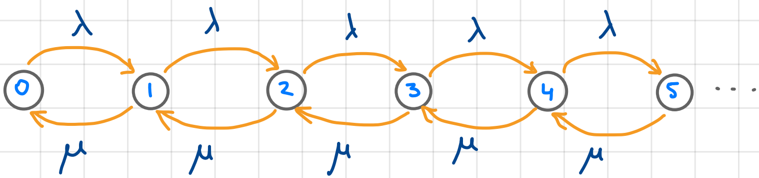
- if $\sum_{i=0}^{\infty} \rho_i < \infty$, we need to find the minimal solution (Thm 7.0)

  which is achieved when $h(0) - h(1) =$

Then $h(1) =$ and (X_t) is transient.

Example: M/M/1 queueing system

Consider the birth and death chain with and (constant and non-zero). Model for a system where jobs (customers) arrive at Poissonian times (at rate λ), queue up, and are executed (served) in the order they arrived at rate μ.



The process (X_t) is the number of jobs (customers) in the queue at time t. From the previous example

$p_i =$, and thus $\sum_{i=0}^{\infty} \left(\frac{\mu}{\lambda}\right)^i = \infty$ if , $\sum_{i=0}^{\infty} \left(\frac{\mu}{\lambda}\right)^i < \infty$ if

Stationary distribution

Def 22.1 Let (X_t) be a continuous-time MC with transition rates $q(i,j)$. A probability distribution π is called stationary (or invariant) if for each state j

Or in terms of the infinitesimal generator

Remark If (Y_n) is the corresponding embedded jump chain, then the stationary distributions for (X_t) and (Y_n) are not the same (and do not necessarily exist simultaneously).

Set $\tilde{\pi}(i) =$. Then $\tilde{\pi}(j) =$

It may be that $\sum \pi(i) < \infty$, but $\sum \tilde{\pi}(i) = \infty$.

Stationary distribution

Let $(X_t)_{t\geq 0}$ be a continuous time non-explosive MC, and suppose that π is a stationary distribution for (X_t).

If $\mathbb{P}[X_0=j] = \pi(j)$ for all states j, then

Proof (for finite state space) Fix state j.

$$\frac{d}{dt}\mathbb{P}[X_t=j]=$$

By Kolmogorov's backward equation

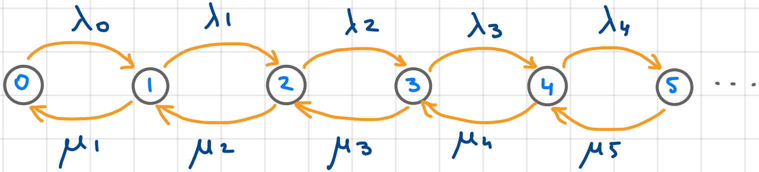
$$\frac{d}{dt}p_t(i,j) = \sum_{k\neq i} q(i,k)\,p_t(k,j) - q(i)\,p_t(i,j)$$

$$\sum_i \pi(i)\frac{d}{dt}p_t(i,j) =$$

$$=$$

Example: Irreducible birth and death chain

Let (X_t) be an irreducible birth and death chain.



$\lambda_i > 0$

$\mu_i > 0$

Equations: $\quad \lambda_0 \pi(0) = \mu_1 \pi(1)$, $\quad (\lambda_j + \mu_j)\pi(j) = \mu_{j+1}\pi(j+1) + \lambda_{j-1}\pi(j-1)$

Rewrite $\left\{ \quad \right.$

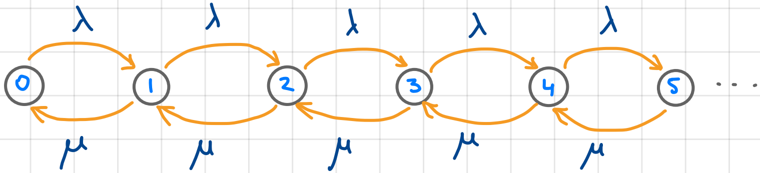
$j=1:$

$\vdots$

Set $\theta_0 = 1$. Then for all j, and the stationary distribution exists iff , in which case

Example: M/M/1 queue

Let (X_t) be an M/M/1 queue



From the previous example

$$\Theta_j = \frac{\lambda_{j-1} \cdots \lambda_0}{\mu_j \cdots \mu_1} =$$

The stationary distribution exists iff $\displaystyle\sum_{j=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^j < \infty$

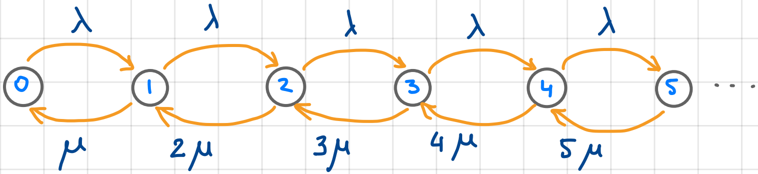
$\displaystyle\sum_{j=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^j < \infty$ iff in which case $\displaystyle\sum_{j=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^j =$

and $\pi(0) =$

$\pi(j) =$

Example: M/M/∞ queue

Queue with infinitely many servers



$\lambda_i = \lambda$

$\mu_i = i\mu$

Repeating the same argument as in the previous example

$$\theta_j = \frac{\lambda_{j-1} \cdots \lambda_0}{\mu_j \cdots \mu_1} =$$

$\sum_{i=0}^{\infty} \theta_i =$ for all $\lambda > 0, \mu > 0$, so the stationary distribution always exists

$\pi(0) =$, $\pi(j) =$

Convergence to the stationary distribution

The exact analog of the convergence theorems for discrete time MC (Cor. 11.1, Thm 11.3, Thm 12.1)

Thm 22.8 Let (X_t) be an irreducible, continuous time MC with transition rates $q(i,j)$. Then TFAE:

(1) All states are positive recurrent

(2) Some state is positive recurrent

(3) The chain is non-explosive and there exists a stationary distribution π.

Moreover, when these conditions hold, the stationary distribution is given by , where T_j is the return time to j;

and for any states i,j.

Convergence to the stationary distribution

<u>Remark</u> There is no issue with periodicity: if $p_t(i,j)>0$ for some $t>0$, then $p_t(i,j)>0$ for all $t>0$

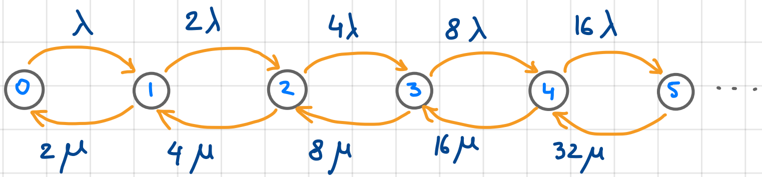
Example: M/M/1 queue is positive recurrent if

null recurrent if

transient if

M/M/∞ queue is always positive recurrent

Example:



$$\theta_j = \frac{\lambda_0 \cdots \lambda_{j-1}}{\mu_1 \cdots \mu_j} =$$

If $\frac{\lambda}{\mu} \in (1,2)$, then

but