MATH 285: Stochastic Processes

math-old.ucsd.edu/~ynemish/teaching/285

Today: Martingales.
Doob's maximal inequality

- Homework 6 is due on Friday, March 4, 11:59 PM

Martingales

Def 24.1 A discrete-time martingale is a stochastic process $(X_n)_{n\geq 0}$ which satisfies $\mathbb{E}[|X_n|]<\infty$ and

$$\mathbb{E}[X_{n+1} \mid X_{0,\dots,} X_n] = X_n \quad \text{for all } n\geq 0$$

Thm 24.8 (Optional sampling theorem)

Let $(X_n)_{n\geq 0}$ be a martingale, and let T be a finite stopping time. Suppose that either

(1) T is bounded: $\exists\, N<\infty$ s.t. $\mathbb{P}[T\leq N]=1$; or

(2) $(X_n)_{0\leq n\leq T}$ is bounded: $\exists\, B<\infty$ s.t. $\mathbb{P}[\,|X_n|\leq B \text{ for all } n\leq T]=1$

Then $\mathbb{E}[X_T]=\mathbb{E}[X_0]$.

Example

Example 25.1 Let (X_n) be a SSRW on $\mathbb{Z}$ conditioned to start at $X_0 = j$ for some $j \in \{0, \dots, N\}$. (X_n) is a martingale.

Denote $\tau_k := \min\{n : X_n = k\}$ $T = \min\{\tau_0, \tau_N\}$ (stopping times).

We computed $\mathbb{P}[\tau_0 < \tau_N]$ using the first-step analysis.

Another approach: use the optional sampling theorem.

- (X_n) is a martingale
- $0 \le X_n \le N$ for all $0 \le n \le T$

By the Optional sampling theorem $\mathbb{E}[X_T] = \mathbb{E}[X_0] = j$

X_T takes two values, so $\mathbb{E}[X_T] = 0 \cdot \mathbb{P}[X_T = 0] + N \cdot \mathbb{P}[X_T = N]$

so $\mathbb{P}[X_T = N] = \frac{j}{N}$, $\mathbb{P}[X_T = 0) = 1 - \frac{j}{N}$. Finally,

$\mathbb{P}[X_T = N] = \mathbb{P}[\tau_N < \tau_0]$, $\mathbb{P}[X_T = 0] = \mathbb{P}[\tau_0 < \tau_N]$

Example

Let $X_1, \dots, X_n, \dots$ be a sequence of i.i.d. random variables with $\mathbb{E}[|X_n|] < \infty$, $\mathbb{E}[X_n] = \mu$ for all n, and denote $S_n := X_1 + \cdots + X_n$

and $M_n := S_n - n \cdot \mu$, $M_0 := 0$

Then $\mathbb{E}[|M_n|] \leq \sum_{i=1}^{n} \mathbb{E}[|X_i|] + n \cdot \mu < \infty$

$$\mathbb{E}[M_{n+1} | M_0, \dots, M_n] = \mathbb{E}[M_n + X_{n+1} - \mu | M_0, \dots, M_n] \qquad n \geq 1$$

$$= M_n + \mathbb{E}[X_{n+1} - \mu] = M_n$$

$$\mathbb{E}[M_1 | M_0] = \mathbb{E}[M_1] = 0 = M_0$$

. (M_n) is a martingale.

Let T be a bounded stopping time for (X_n) (and for (M_n)).

Then by the Optional sampling theorem

$$0 = \mathbb{E}[M_0] = \mathbb{E}[M_T] = \mathbb{E}[S_T - T\mu] = \mathbb{E}\Big[\sum_{i=1}^{T} X_i\Big] - \mu\,\mathbb{E}[T]$$

Therefore, $\mathbb{E}\Big[\sum_{i=1}^{T} X_i\Big] = \mu\,\mathbb{E}[T]$

Submartingales / supermartingales

$\mathbb{E}[|X_n|] < \infty$

A stochastic process (X_n) is called

a submartingale if $\mathbb{E}[X_{n+1} \mid X_0, \dots, X_n] \geq X_n$ for all n

a supermartingale if $\mathbb{E}[X_{n+1} \mid X_0, \dots, X_n] \leq X_n$ for all n

We use (sub) martingales to establish the maximal inequalities. Recall the Markov's inequality : $\forall a > 0$

$$\mathbb{P}[|X| \geq a] \leq \frac{\mathbb{E}[|X|]}{a}$$

In particular, if (X_n) is a submartingale and $X_n \geq 0$, then for any $i \leq n$

$$\mathbb{P}[X_i \geq a] \leq \frac{\mathbb{E}[X_i]}{a} \leq \frac{\mathbb{E}[X_n]}{a}$$

In fact a stronger statement holds.

Doob's maximal inequality

Thm 25.3 Let (X_n) be a non-negative submartingale.

Then for any $a>0$

$$\mathbb{P}[\max\{X_0,\dots,X_n\}\geq a]\leq \frac{\mathbb{E}[X_n]}{a}$$

Proof. Let $T:=\min\{n: X_n\geq a\}$, a stopping time.

- $A_k := \{T=k\} = \{T\leq k\}\setminus\{T\leq k-1\}$ is $(X_0,\dots,X_k)$-measurable
- Since $X_n\geq 0$, $\mathbb{E}[X_n]\geq \mathbb{E}[X_n \mathbb{1}_{\{T\leq n\}}] = \sum_{k=0}^{n}\mathbb{E}[X_n \mathbb{1}_{\{T=k\}}]$
- $\mathbb{E}[X_n \mathbb{1}_{A_k}] = \mathbb{E}[\mathbb{E}[X_n \mathbb{1}_{A_k} \mid X_0,\dots,X_k]] = \mathbb{E}[\mathbb{1}_{A_k}\mathbb{E}[X_n\mid X_0,\dots,X_k]]$

  $\overset{SM}{\geq} \mathbb{E}[\mathbb{1}_{A_k} X_k] = \mathbb{E}[\mathbb{1}_{\{T=k\}} X_k] \geq \mathbb{E}[\mathbb{1}_{\{T=k\}}a] = a\,\mathbb{P}[T=k]$
- $\mathbb{E}[X_n]\geq \sum_{k=0}^{n} a\,\mathbb{P}[T=k] = a\cdot\mathbb{P}[T\leq n]$
- $\mathbb{P}[T\leq n] = \mathbb{P}[\max\{X_0,\dots,X_n\}\geq a]$ ∎

Doob's maximal inequality

Lemma 25.4 Let (X_n) be a martingale, and let $f: \mathbb{R} \to \mathbb{R}$ be a convex function such that $\mathbb{E}[|f(X_n)|] < \infty$ for all n.

Then $Y_n = f(X_n)$ is a submartingale.

Proof Exercise.

Corollary 25.5 Let (X_n) be a martingale, let $r \geq 1$, $a, b \geq 0$.

Then

(i) $$\mathbb{P}[\max\{X_0, \dots, X_n\} \geq a] \leq \frac{\mathbb{E}[|X_n|^r]}{a^r}$$

(ii) $$\mathbb{P}[\max\{X_0, \dots, X_n\} \geq a] \leq \frac{\mathbb{E}[e^{bX_n}]}{e^{ba}}$$

Proof. If $r \geq 1$, then $f(x) = |x|^r$ is a convex function.

By Lemma 25.4 $(|X_n|^r)$ is a non-negative submartingale.

Doob's maximal inequality

Fix $a>0$. If $X_k \geq a$, then $|X_k|^r \geq a^r$. Therefore,

$$\mathbb{P}[\max\{X_0,\dots,X_n\} \geq a] \leq \mathbb{P}[\max\{|X_0|^r,\dots,|X_n|^r\} \geq a^r]$$
$$\leq \frac{\mathbb{E}[|X_n|^r]}{a^r}$$

The second inequality is proven using a similar argument. ■

Example 25.6 Let $X_1, X_2, \dots$ be i.i.d. symmetric Bernoulli random variables, $S_0 = 0$, $S_n = X_1 + \cdots + X_n$. (S_n) is a martingale.

Take (ii) in Corollary 25.5 with $b = \frac{1}{\sqrt{n}}$ $a = \alpha\sqrt{n}$, so that

$$\mathbb{P}[\max\{S_0,\dots,S_n\} \geq \alpha\sqrt{n}] \leq e^{-ba}\,\mathbb{E}[e^{bS_n}] = e^{-\alpha}\,\mathbb{E}[e^{S_n/\sqrt{n}}]$$

Now $\mathbb{E}[e^{S_n/\sqrt{n}}] = \left(\mathbb{E}[e^{X_1/\sqrt{n}}]\right)^n = \left(\frac{1}{2}\left(e^{\frac{1}{\sqrt{n}}} + e^{-\frac{1}{\sqrt{n}}}\right)\right)^n \to e^{1/2}, n \to \infty \Rightarrow \mathbb{E}[e^{S_n/\sqrt{n}}] \leq C$

Therefore, for any n $\mathbb{P}[\max\{S_0,\dots,S_n\} \geq \alpha\sqrt{n}] \leq e^{-\alpha} \cdot C$ ≈ 0.01