MATH 285: Stochastic Processes

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Today: Martingales convergence theorem

- Homework 7 is due on Friday, March 11, 11:59 PM

The martingale convergence theorem

Theorem 26.1 Let $(X_n)_{n\geq 0}$ be a martingale, and suppose there exists $C \geq 0$ such that $\mathbb{P}[X_n \geq -C] = 1$ for all n.

Then there is a random variable X_∞ such that

Proof (1) Enough to prove for $C = 0$

Consider $Y_n =$. Then (Y_n) is a martingale, $Y_n \geq 0$,

and if and only if

Assume that

(2)

- (X_n) is a nonnegative martingale, therefore by

The martingale convergence theorem

- Doob's Maximal inequality for any $N \in \mathbb{N}$

$$\mathbb{P}\left[\max_{0 \le n \le N} X_n \ge a\right] \le$$

- Take the limit $N \to \infty$ (monotonicity of $\mathbb{P}$)

$$\lim_{N\to\infty} \mathbb{P}\left[\max_{0 \le n \le N} X_n \ge a\right] =$$

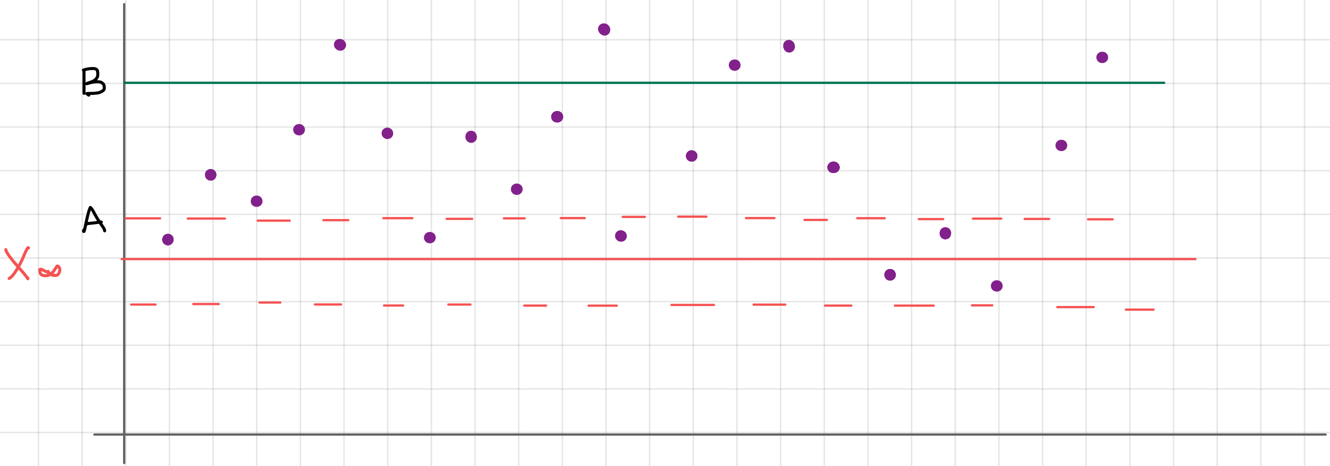
- Take the limit $a \to \infty$

$$\lim_{a\to\infty} \mathbb{P}\left[\max_{n \ge 0} X_n \le a\right] =$$

(3) Each trajectory $(X_n(\omega))$ has a convergent subsequence $(X_{n_k}(\omega))$, denote the limit $X_\infty(\omega)$

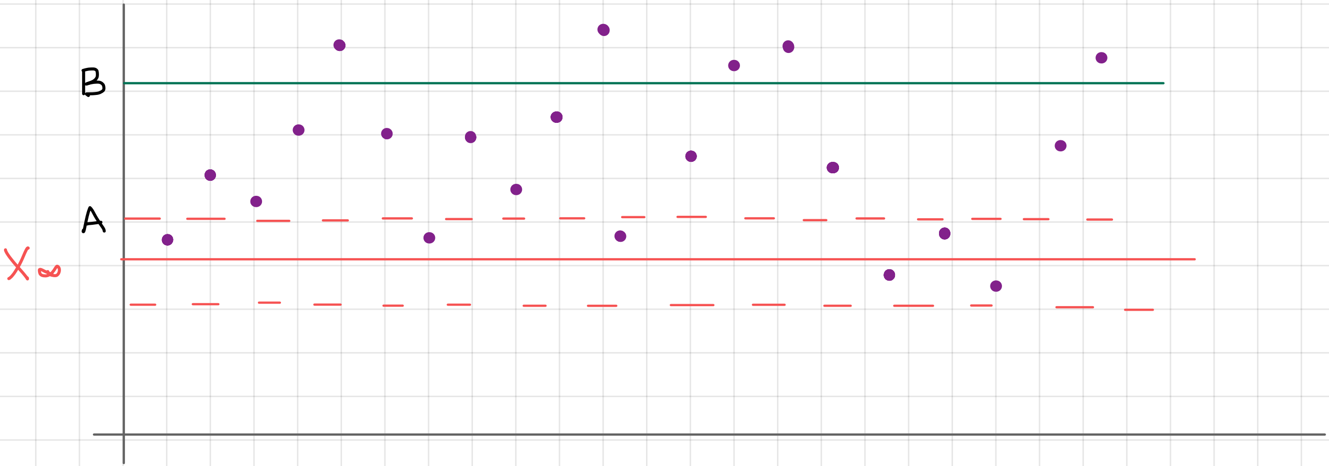
The martingale convergence theorem

(4) If $(X_n(\omega))$ is not convergent, there are infinitely many terms $X_n(\omega)$ away from $X_\infty(\omega)$



If $(X_n(\omega))$ is not convergent, there are $A, B \in \mathbb{Q}$, $A, B \geq 0$, $A < B$ such that there are infinitely many terms $X_n(\omega) \geq B$ and infinitely many terms $X_n(\omega) \leq A$.

The martingale convergence theorem



For any $A, B \in \mathbb{Q}$, $A, B \geq 0$, $A < B$ denote

$S_1 =$, $T_n =$, $S_n =$

(S_n, T_n) denotes an (A,B)-upcrossing

(5) If $(X_n(\omega))$ is not convergent, then there exist infinitely many (A,B) upcrossings for some $A < B$

The martingale convergence theorem

Fix A, B. Denote , number of (A, B)-upcrossings before time n.

Denote , total number of (A, B)-upcrossings.

(6)

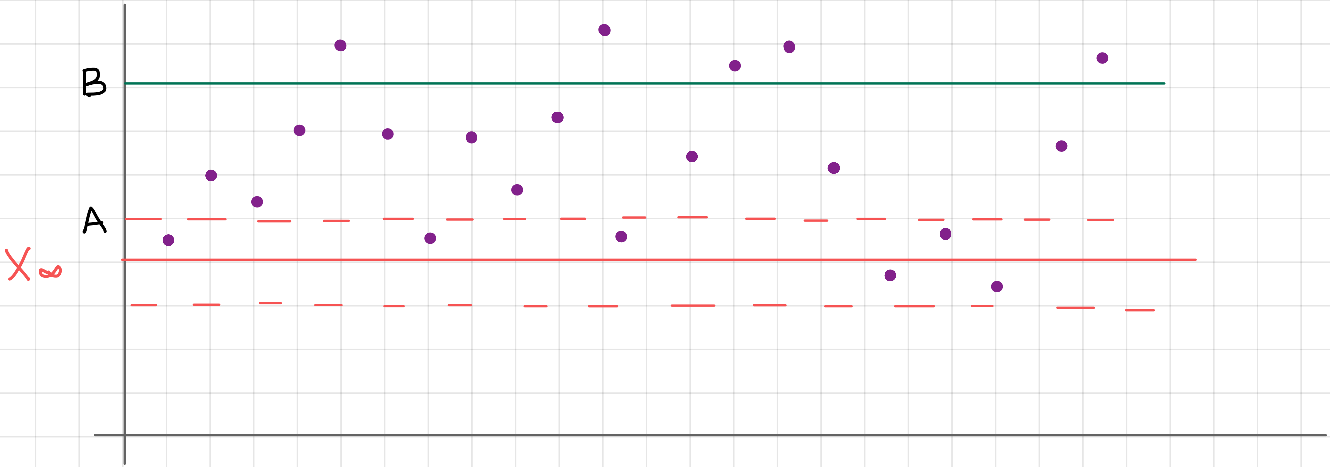
- Consider the following game:

  bet $B_j = \Bigg\{$, win/lose

  $\hookleftarrow$ $(X_0, \dots, X_{j-1})$-measurable

  Total winnings: $W_n =$

The martingale convergence theorem



- $(W_n)_{n\geq 1}$ is a martingale, therefore

  : $W_1 = \Big\{$,

- $W_n =$

The martingale convergence theorem

- $\mathbb{E}[W_n] = 0 \geq (b-a)\,\mathbb{E}[U_n] - a \quad \Rightarrow$

- $\lim\limits_{n\to\infty} \mathbb{E}[U_n] = \qquad\qquad \Rightarrow$

(7) For any $A, B \in \mathbb{Q}$, $A, B \geq 0$, $A < B$

$$\mathbb{P}[\text{ infinitely many } (A,B)\text{-upcrossings}] = 0$$

(8) $\mathbb{P}[\exists\, A, B \in \mathbb{Q},\ A, B \geq 0,\ A < B$ s.t. the exists ∞-many (A,B)-upcrossings$] = 0$

$\blacksquare$

Example

$(X_n)_{n\geq 0}$ SSRW on $\mathbb{Z}$, $X_0 =$. $T =$

Consider $M_n :=$. M_n is a nonnegative martingale.

Therefore, by the Martingale convergence thm there exists r.v. M_∞ s.t. $\mathbb{P}\left[\lim_{n\to\infty} M_n = M_\infty\right] = 1$.

What is M_∞? $M_n(\omega)$ is eventually constant for any ω.

Since $\{M_n(\omega) = k, M_{n+1}(\omega) = k\}$ is not possible for any $k \geq 1$, $M_\infty = 0$ with probability 1.

Remark $\mathbb{E}[M_n] = \mathbb{E}[M_0] = \mathbb{E}[X_0] = 1$, but $M_\infty = 0$.

In particular,

Example. Poya Urns

An urn initially contains a red balls and b blue balls. At each step, draw a ball uniformly at random and return it with another ball of the same color. Denote by X_n the number of red balls in the urn after n turns. Then (X_n) is a Markov chain (time inhomogeneous)

$$\mathbb{P}[X_{n+1} = k+1 \mid X_n = k] = \qquad\qquad , \quad \mathbb{P}[X_{n+1} = k \mid X_n = k] =$$

Long-run behavior of the process? Techniques developed for time-homogeneous MC cannot be applied.

Let $M_n :=$ be the fraction of red ball after n turns.

Then ,

Example. Poya Urns

Next, $\mathbb{E}[X_{n+1} \mid X_0, \dots, X_n] =$ (X_n is Markov)

and $\mathbb{E}[X_{n+1} \mid X_n] =$

$=$

$\mathbb{E}[M_{n+1} \mid M_0, \dots, M_n] =$

(M_n) is a nonnegative martingale. Therefore, by the Martingale convergence theorem $M_n \to M_\infty$, $n \to \infty$ a.s.

One can show that M_∞ has beta distribution

$$f_{M_\infty}(x) = \frac{(a+b-1)!}{(a-1)!\,(b-1)!} x^{a-1} (1-x)^{b-1}, \quad 0 < x < 1$$