MATH 285: Stochastic Processes

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Today: Transition probabilities.
Hitting times

- Test Homework on Gradescope

Last time

Def 1.5 Let X_n be a discrete time stochastic process with state space S that is finite or countably infinite. Then X_n is called a discrete time Markov chain if for each $n \in \mathbb{N}$ and each $(i_1, \dots, i_n) \in S^n$

(M) $$\mathbb{P}[X_n = i_n \mid X_1 = i_1, \dots, X_{n-1} = i_{n-1}] = \mathbb{P}[X_n = i_n \mid X_{n-1} = i_{n-1}]$$

Example 1.2 (Recall $\{X_i\}$ are i.i.d.)

Suppose that S is finite or countably infinite

Then (by independence) $\mathbb{P}[X_n = i_n \mid X_1 = i_1, \dots, X_{n-1} = i_{n-1}] = \mathbb{P}[X_n = i_n]$ and $\mathbb{P}[X_n = i_n \mid X_{n-1} = i_{n-1}] = \mathbb{P}[X_n = i_n]$, so (M) is satisfied.

$$\mathbb{P}[X_1 = i_1, \dots, X_n = i_n] = \mathbb{P}[X_1 = i_1, \dots, X_n = i_n]$$

Discrete time Markov chain

Exemple 1.2 (cont.) Recall $S_n = X_1 + \cdots + X_n$, so $X_n = S_n - S_{n-1}$

and thus $\mathbb{P}[S_1 = i_1, \dots, S_n = i_n] = \mathbb{P}[X_1 = i_1, X_2 = i_2 - i_1, \dots, X_n = i_n - i_{n-1}]$

Check (M)

$$\mathbb{P}[S_n = i_n \mid S_1 = i_1, \dots, S_{n-1} = i_{n-1}] = \frac{\mathbb{P}[S_1 = i_1, S_2 = i_2, \dots, S_n = i_n]}{\mathbb{P}[S_1 = i_1, S_2 = i_2, \dots, S_{n-1} = i_{n-1}]}$$

$$= \frac{\mathbb{P}[X_1 = i_1, X_2 = i_2 - i_1, \dots, X_{n-1} = i_{n-1} - i_{n-2}, X_n = i_n - i_{n-1}]}{\mathbb{P}[X_1 = i_1, X_2 = i_2 - i_1, \dots, X_{n-1} = i_{n-1} - i_{n-2}]}$$

$$= \mathbb{P}[X_n = i_n - i_{n-1}]$$

$\{S_{n-1} = i_{n-1}, S_{n-1} + X_n = i_n\} = \{S_{n-1} = i_{n-1}, X_n = i_n - i_{n-1}\}$

$$\mathbb{P}[S_n = i_n \mid S_{n-1} = i_{n-1}] = \frac{\mathbb{P}[S_{n-1} = i_{n-1}, S_n = i_n]}{\mathbb{P}[S_{n-1} = i_{n-1}]} = \frac{\mathbb{P}[S_{n-1} = i_{n-1}, X_n = i_n - i_{n-1}]}{\mathbb{P}[S_{n-1} = i_{n-1}]}$$

$$= \mathbb{P}[X_n = i_n - i_{n-1}]$$

We conclude that S_n is also a Markov chain.

Transition probabilities. Time-homogeneous MC

"Distribution" of a Markov chain is completely described by the collection $\{ \mathbb{P}[X_n = j \mid X_{n-1} = i] : (i,j) \in S^2, n \in \mathbb{N} \}$

Def. 1.6 A Markov chain is called time-homogeneous if for any $i,j \in S$ $\mathbb{P}[X_n = j \mid X_{n-1} = i]$ is independent of n

i.e., there exists a function $p: S \times S \to [0,1]$ s.t.

$$\mathbb{P}[X_n = j \mid X_{n-1} = i] = p(i,j) \quad \forall n \in \mathbb{N}$$

We call $\mathbb{P}[X_n = j \mid X_{n-1} = i]$ the transition probabilities

"Distribution" of a time-homogeneous MC is determined by the transition probabilities $p(i,j)$ and the distribution of X_1 (initial distribution)

Transition probabilities

If $p(i,j)$ are the transition probabilities, then

$$\sum_{j\in S} p(i,j) = \sum_{j\in S} \mathbb{P}[X_2 = j \mid X_1 = i] = \sum_{j\in S} \frac{\mathbb{P}[X_2 = j, X_1 = i]}{\mathbb{P}[X_1 = i]} = \frac{\mathbb{P}[X_1 = i]}{\mathbb{P}[X_1 = i]} = 1$$

Def. If A is an $n\times n$ matrix s.t. $\forall i \in \{1,\dots,n\}$

$$\sum_{j=1}^{n} A_{ij} = 1,$$ then A is called **stochastic**

Suppose $|S| < \infty$ and let

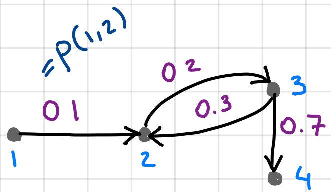
$$P := [p(i,j)]_{i,j\in S}, \quad P = \begin{array}{c|ccc} & s_1 & s_2 & \cdots \\ \hline s_1 & p(s_1,s_1) & p(s_1,s_2) & \cdots \\ s_2 & p(s_2,s_1) & p(s_2,s_2) & \cdots \\ \vdots & \vdots & & \end{array}$$

transition matrix

Then P is stochastic

Ex.

$S = \{1, 2, 3, 4\}$



$$P = [p(i,j)] = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0.9 & 0.1 & 0 & 0 \\ 2 & 0 & 0.8 & 0.2 & 0 \\ 3 & 0 & 0.3 & 0 & 0.7 \\ 4 & 0 & 0 & 0 & 1 \end{array}$$

Transition probabilities

Example 1.7

Markov chain on $S = \{0, 1, 2, \dots, N\}$

Transition probabilities:

if $i \in \{1, 2, \dots, N-1\}$ then $p(i,j) = \begin{cases} \frac{1}{2}, & \text{if } j = i+1 \\ \frac{1}{2}, & \text{if } j = i-1 \\ 0, & \text{otherwise} \end{cases}$

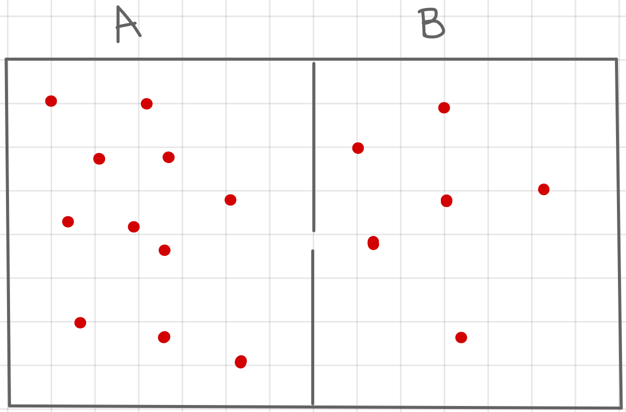
Reflecting random walk: $p(0,1) = p(N, N-1) = 1$

Absorbing random walk: $p(0,0) = p(N,N) = 1$

Partially reflecting walk: $p(0,0) = p \quad p(0,1) = 1-p$

$p(N,N) = q \quad p(N,N-1) = 1-q$

Ehrenfest model



Diffusion through a membrane:

N particles in two chambers connected by a small hole

[Think about two urns with N balls]

At each step you choose (unif. at random) a ball and put it to the other urn [particle passing through the membrane]. Denote $X_n = \#$ balls in urn A. $X_n \in \{0, 1, \ldots, N\}$

(X_n) is a Markov chain

$$\forall i \in \{0, 1, \ldots, N\} \quad p(i, i+1) = \frac{N-i}{N}$$
$$p(i, i-1) = \frac{i}{N}$$

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & \cdots & N \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & & \\ \frac{1}{N} & 0 & \frac{N-1}{N} & 0 & \cdots & \\ 0 & \frac{2}{N} & 0 & \frac{N-2}{N} & \cdots & \\ & & & & & \\ & & & & & \end{bmatrix} \end{array}$$

n-step transition probabilities

Let $(X_n)_{n \in \mathbb{N} \cup \{0\}}$ be a Markov chain (time-homogeneous, finite S)

Q: Given X_0, what is the distribution of X_n?

Lemma 2.3 Let (X_n) be a time-homogeneous Markov chain with a finite state space S and transition matrix P.

Then $\forall n \in \mathbb{N}$ $\quad \mathbb{P}[X_n = j \mid X_0 = i] = [P^n]_{ij}$

Proof (Induction by n) $n=1$: $\mathbb{P}[X_1 = j \mid X_0 = i] = P_{ij}$

Induction step: Suppose that $\mathbb{P}[X_n = j \mid X_0 = i] = [P^n]_{ij}$. Then

$$\begin{aligned}
\mathbb{P}[X_{n+1} = j \mid X_0 = i] &= \sum_{k \in S} \mathbb{P}[X_{n+1} = j, X_n = k \mid X_0 = i] \\
&= \sum_{k \in S} \mathbb{P}[X_{n+1} = j \mid X_n = k, X_0 = i]\, \mathbb{P}[X_n = k \mid X_0 = i] \\
&= \sum_{k \in S} \mathbb{P}[X_{n+1} = j \mid X_n = k]\, [P^n]_{ik} = \sum_k [P^n]_{ik} P_{kj} \\
&= [P^{n+1}]_{ij} \quad \blacksquare
\end{aligned}$$

Chapman - Kolmogorov Equations

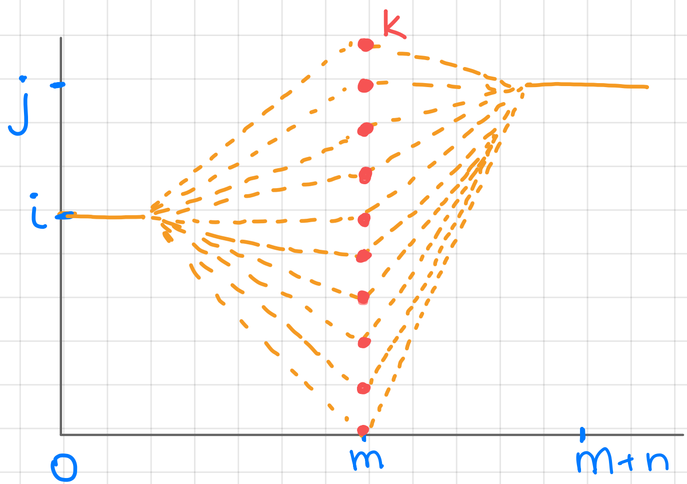
Cor. 2.4 Let (X_n) be as in Lem 2.3. Denote by $p_n(i,j)$ the n-step transition probability $p_n(i,j) = \mathbb{P}[X_n = j \mid X_0 = i]$. Then for any $m, n \in \mathbb{N}$

$$p_{m+n}(i,j) = \sum_{k \in S} p_m(i,k)\, p_n(k,j)$$

Proof. By Lem 2.3, $\forall n \in \mathbb{N}$ $p_n(i,j) = [P^n]_{ij}$. Then

$$p_{m+n}(i,j) = [P^{m+n}]_{ij} = \sum_k P^m_{ik} P^n_{kj}$$

$$= \sum_k p_m(i,k)\, p_n(k,j) \qquad \blacksquare$$



Markov property

"future is independent of the past"

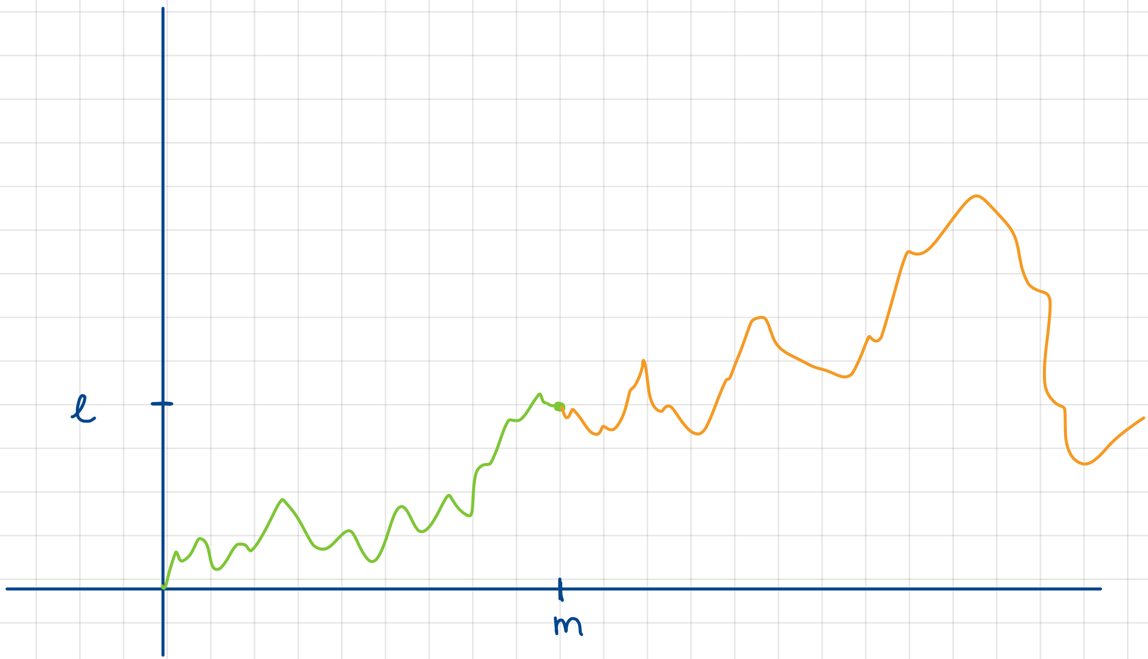
Prop 2.5 Let (X_n) be a time-homogeneous MC with discrete state space S and transition probabilities $p(i,j)$. Fix $m \in \mathbb{N}$, $\ell \in S$, and suppose that $\mathbb{P}[X_m = \ell] > 0$. Then conditional on $X_m = \ell$, the process $(X_{m+n})_{n \in \mathbb{N} \cup \{0\}}$ is Markov with transition probabilities $p(i,j)$ initial distribution δ_ℓ (atom at ℓ) and independent of the random variables $X_0, X_1, \ldots X_m$

Proof Let A be an event determined by $X_0, \ldots, X_m$.

- First assume that for some $(i_0, \ldots, i_m) \in S^{m+1}$

Then $\mathbb{P}[\{X_m = i_m, \cdots, X_{m+n} = i_{m+n}\} \cap A \mid X_m = \ell]$

$=$



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