MATH 285: Stochastic Processes

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Today: Transition probabilities.
Hitting times

- Test Homework on Gradescope

Last time

<u>Def 1.5</u> Let X_n be a discrete time stochastic process with state space S that is finite or countably infinite. Then X_n is called a discrete time Markov chain if for each $n \in \mathbb{N}$ and each $(i_1, \dots, i_n) \in S^n$

$$(M) \qquad \mathbb{P}[X_n = i_n \mid X_1 = i_1, \dots, X_{n-1} = i_{n-1}] = \mathbb{P}[X_n = i_n \mid X_{n-1} = i_{n-1}]$$

<u>Example 1.2</u> (Recall $\{X_i\}$ are i.i.d.)

Suppose that S is finite or countably infinite

Then (by independence) $\mathbb{P}[X_n = i_n \mid X_1 = i_1, \dots, X_{n-1} = i_{n-1}] = \mathbb{P}[X_n = i_n]$ and $\mathbb{P}[X_n = i_n \mid X_{n-1} = i_{n-1}] = \mathbb{P}[X_n = i_n]$, so (M) is satisfied.

$$\mathbb{P}[X_1 = i_1, \dots, X_n = i_n] = \mathbb{P}[X_1 = i_1, \dots, X_n = i_n]$$

Discrete time Markov chain

Exemple 1.2 (cont.) Recall $S_n = X_1 + \cdots + X_n$, so $X_n =$

and thus $\mathbb{P}[S_1 = i_1, \ldots, S_n = i_n] =$

Check (M)

$$\mathbb{P}[S_n = i_n \mid S_1 = i_1, \ldots, S_{n-1} = i_{n-1}] = \frac{\mathbb{P}[S_1 = i_1, S_2 = i_2, \ldots, S_n = i_n]}{\mathbb{P}[S_1 = i_1, S_2 = i_2, \ldots, S_{n-1} = i_{n-1}]}$$

$=$

$=$

$\mathbb{P}[S_n = i_n \mid S_{n-1} = i_n] =$

$=$

We conclude that S_n is

Transition probabilities. Time-homogeneous MC

"Distribution" of a Markov chain is completely described by the collection

Def. 1.6 A Markov chain is called time-homogeneous if for any $i, j \in S$

i.e., there exists a function $p : S \times S \to [0,1]$ s.t.

We call $\mathbb{P}[X_n = j \mid X_{n-1} = i]$ the

"Distribution" of a time-homogeneous MC is determined by the

and

Transition probabilities

If $p(i,j)$ are the transition probabilities, then

$$\sum_{j \in S} p(i,j) =$$

<u>Def</u>. If A is an $n \times n$ matrix s.t. $\forall i \in \{1, \ldots, n\}$

$$\sum_{j=1}^{n} A_{ij} = 1,$$ then A is called

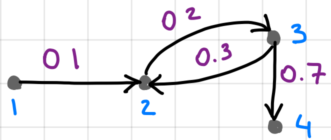
Suppose $|S| < \infty$ and let

$$P := [p(i,j)]_{i,j \in S}, \quad P = \begin{array}{cc} & \begin{array}{ccc} s_1 & s_2 & \cdots \end{array} \\ \begin{array}{c} s_1 \\ s_2 \\ \vdots \end{array} & \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \end{array}$$

Then

<u>Ex.</u>

$$S = \{1, 2, 3, 4\}$$



$$P = [p(i,j)] = \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \end{array}$$

Transition probabilities

Example 1.7

Markov chain on $S = \{0, 1, 2, \dots, N\}$

Transition probabilities:

if $i \in \{1, 2, \dots, N-1\}$ then $p(i,j) = \left\{ \vphantom{\begin{matrix} a \\ b \\ c \end{matrix}} \right.$

Reflecting random walk:

Absorbing random walk:

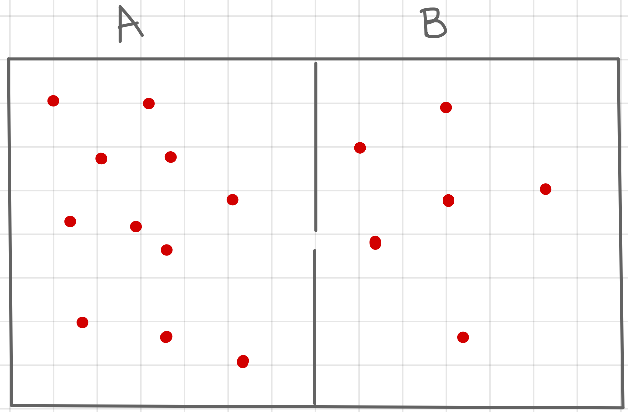
Partially reflecting walk:

Ehrenfest model

Diffusion through a membrane:

N particles in two chambers connected by a small hole

[Think about two urns with N balls]



At each step you choose (unif. at random) a ball and put it to the other urn [particle passing through the membrane]. Denote $X_n = \#$ balls in urn A. $X_n \in \{0, 1, \dots, N\}$

(X_n) is a Markov chain

$\forall i \in \{0, 1, \dots, N\}$ $\quad p(i, i+1) =$

$\qquad\qquad\qquad\qquad\;\; p(i, i-1) =$

$$P = \begin{array}{cc} & \begin{matrix} 0 & 1 & 2 & 3 & \cdots & N \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{matrix} & \begin{bmatrix} \; & & & & & \; \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix} \end{array}$$

n-step transition probabilities

Let $(X_n)_{n\in\mathbb{N}\cup\{0\}}$ be a Markov chain (time-homogeneous, finite S)

Q: Given X_0, what is the distribution of X_n?

Lemma 2.3 Let (X_n) be a time-homogeneous Markov chain with a finite state space S and transition matrix P.

Then $\forall n \in \mathbb{N}$

Proof (Induction by n) $n=1$: $\mathbb{P}[X_1=j \mid X_0=i] = P_{ij}$

Induction step: Suppose that $\mathbb{P}[X_n=j \mid X_0=i] = [P^n]_{ij}$. Then

$$\mathbb{P}[X_{n+1}=j \mid X_0=i] =$$

$$=$$

$$=$$

$$=$$

$\blacksquare$

Chapman - Kolmogorov Equations

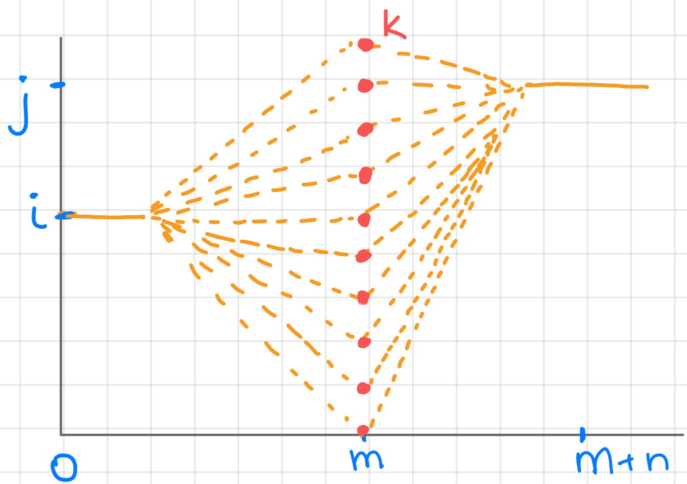
Cor. 2.4 Let (X_n) be as in Lem 2.3. Denote by $p_n(i,j)$ the n-step transition probability $p_n(i,j) = \mathbb{P}[X_n = j \mid X_0 = i]$.
Then for any $m, n \in \mathbb{N}$

$$p_{m+n}(i,j) =$$

Proof. By Lem 2.3, $\forall n \in \mathbb{N}$ $p_n(i,j) = [P^n]_{ij}$. Then

$$p_{m+n}(i,j) = [P^{m+n}]_{ij} = \sum_k P^m_{ik} P^n_{kj}$$

$$= \sum_k p_m(i,k)\, p_n(k,j) \qquad \blacksquare$$



Markov property   "future is independent of the past"

Prop 2.5 Let (X_n) be a time-homogeneous MC with discrete state space S and transition probabilities $p(i,j)$. Fix $m \in \mathbb{N}$, $\ell \in S$, and suppose that $\mathbb{P}[X_m = \ell] > 0$. Then conditional on $X_m = \ell$, the process $(X_{m+n})_{n \in \mathbb{N}}$ is with transition probabilities initial distribution (atom at ℓ) and independent of the random variables

Proof Let A be an event determined by $X_0, \dots, X_m$.

- First assume that for some $(i_0, \dots, i_m) \in S^{m+1}$

Then $\mathbb{P}[\{X_m = i_m, \dots, X_{m+n} = i_{m+n}\} \cap A \mid X_m = \ell]$

$=$

Markov property

- Any set A determined by $X_0, \dots, X_m$ is a disjoint union of the events of the form $\{X_0 = i_0, \dots, X_m = i_m\}$.

E.g. $\mathbb{P}[\{X_m = i_m, \dots, X_{m+n} = i_{m+n}\} \cap (A_1 \sqcup A_2) \mid X_m = \ell]$

So $(*)$ holds for any event A.

Hitting times

Q1: When is the first time the process enters a certain set?

For $A \subset S$, compute

Q2: For $A, B \subset S$, $A \cap B = \emptyset$ find the probability

Start with Q2

- trivial:

- take $i \notin A \cup B$; "first step analysis":

$$\mathbb{P}[\tau_A < \tau_B \mid X_0 = i] =$$

By the Markov property

$$\mathbb{P}[\tau_A < \tau_B \mid X_0 = i, X_1 = j] =$$

Hitting times

We conclude that

$$h(i) = \qquad\qquad\qquad\qquad (**)$$

This gives a system of linear equations + boundary conditions

$$h(i) = \begin{cases} 1, & i \in A \\ 0, & i \in B \end{cases}.$$

If S is finite, denote $\bar{h} := (h(1), h(2), \dots, h(|S|))$. Then

$(**)$ becomes