MATH 285: Stochastic Processes

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Today: Hitting times. First step analysis

- Test Homework on Gradescope

Initial distribution and transition matrix

Let $(X_n)_{n\geq 0}$ be a (time-homogeneous) Markov chain with finite state space $S=\{s_1, s_2, \dots, s_{|S|}\}$ $(=\{1,2,3,\dots,|S|\})$

Distribution of X_n is a vector $(\mathbb{P}[X_n=1], \mathbb{P}[X_n=2], \dots, \mathbb{P}[X_n=|S|])$

Let $\lambda=(\lambda_1, \lambda_2, \dots, \lambda_{|S|})$ be the distribution of X_0, i.e., $\mathbb{P}[X_0=i]=\lambda_i$. Let P be the transition matrix of (X_n).

Q: What is the distribution of X_n?

X_1: $\mathbb{P}[X_1=j] = \sum_{i\in S} \mathbb{P}[X_1=j \mid X_0=i]\, \mathbb{P}[X_0=i] = \sum_{i=1}^{|S|} \lambda_i\, p(i,j) = [\lambda P]_j$

Distribution of X_1 is given by λP

X_n: $\mathbb{P}[X_n=j] = \sum_{i=1}^{|S|} \mathbb{P}[X_n=j \mid X_0=i]\, \mathbb{P}[X_0=i] = \sum_{i=1}^{|S|} \lambda_i\, p_n(i,j) = [\lambda P^n]_j$

Distribution of X_n is given by λP^n

We will say that (X_n) is Markov(λ, P)

Markov property "future is independent of the past"

Prop 2.5 Let (X_n) be a time-homogeneous MC with discrete state space S and transition probabilities $p(i,j)$. Fix $m \in \mathbb{N}$, $\ell \in S$, and suppose that $\mathbb{P}[X_m = \ell] > 0$. Then conditional on $X_m = \ell$, the process $(X_{m+n})_{n\in\mathbb{N}}$ is Markov with transition probabilities $p(i,j)$, initial distribution

$$\begin{array}{ccccccccc} 1 & & \ell-1 & \ell & \ell+1 & & \\ (0, & \dots, & 0, & 1, & 0, & \dots, & 0) \end{array}$$

and independent of the random variables $X_0, \dots, X_m$, i.e. if A is an event determined by $X_0, X_1, \dots, X_m$ and $\mathbb{P}[A \cap \{X_m = \ell\}] > 0$ then for all $n \geq 0$

$$\mathbb{P}[X_{m+1} = i_{m+1}, \dots, X_{m+n} = i_{m+n} \mid A \cap \{X_m = \ell\}] = p(\ell, i_{m+1})\, p(i_{m+1}, i_{m+2}) \cdots p(i_{m+n-1}, i_{m+n})$$

Proof. Enough to show that

$$\mathbb{P}[\{X_{m+1} = i_{m+1}, \dots, X_{m+n} = i_{m+n}, X_m = \ell\} \cap A] \overset{(*)}{=} p(\ell, i_{m+1}) \cdots p(i_{m+n-1}, i_{m+n})\, \mathbb{P}[A \cap \{X_m = \ell\}]$$

Markov property

- Let $A = \{X_0 = i_0, \dots, X_m = \ell\}$. Then

$$\mathbb{P}[X_0 = i_0, \dots, X_m = \ell, X_{m+1} = i_{m+1}, \dots, X_{m+n} = i_{m+n}] = \mathbb{P}[X_0 = i_0]\, p(i_0, i_1) \cdots p(i_{m-1}, \ell) \times$$
$$\times\, p(\ell, i_{m+1}) \cdots p(i_{m+n-1}, i_{m+n})$$

$$\mathbb{P}[X_0 = i_0, \dots, X_m = \ell] = \mathbb{P}[X_0 = i_0]\, p(i_0, i_1) \cdots p(i_{m-1}, \ell)$$

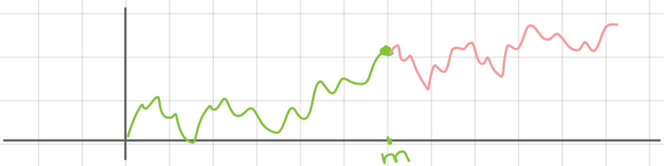
- Any set A determined by $X_0, \dots, X_m$ is a disjoint union of the events of the form $\{X_0 = i_0, \dots, X_m = i_m\}$.

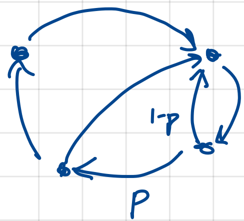
E.g. $\mathbb{P}[\{X_{m+1} = i_{m+1}, \dots, X_{m+n} = i_{m+n}\} \cap (A_1 \sqcup A_2) \cap \{X_m = \ell\}]$

$$= p(\ell, i_{m+1}) \cdots p(i_{m+n-1}, i_{m+n}) \left(\mathbb{P}[A_1 \cap \{X_m = \ell\}] + \mathbb{P}[A_2 \cap \{X_m = \ell\}] \right)$$

$$= p(\ell, i_{m+1}) \cdots p(i_{m+n-1}, i_{m+n})\, \mathbb{P}[(A_1 \cup A_2) \cap \{X_m = \ell\}]$$

So $(*)$ holds for any event A.





Hitting times

Q1: When is the first time the process enters a certain set?

For $A \subset S$, compute $\tau_A := \min\{n \in \mathbb{N} \cup \{0\} : X_n \in A\}$

Q2: For $A, B \subset S$, $A \cap B = \emptyset$ find the probability

$$\mathbb{P}[\tau_A < \tau_B \mid X_0 = i] : h(i)$$

Start with Q2

- trivial: $h(i) = \begin{cases} 1, & i \in A \\ 0, & i \in B \end{cases}$

- take $i \notin A \cup B$; "first step analysis":

$$\mathbb{P}[\tau_A < \tau_B \mid X_0 = i] = \sum_{j \in S} \mathbb{P}[\tau_A < \tau_B \mid X_1 = j, X_0 = i]\, \mathbb{P}[X_1 = j \mid X_0 = i]$$

By the Markov property

$$\mathbb{P}[\tau_A < \tau_B \mid X_0 = i, X_1 = j] = \mathbb{P}[\tau_A < \tau_B \mid X_1 = j] = \mathbb{P}[\tau_A - 1 < \tau_B - 1 \mid X_0 = j] = h(j)$$

$X_k \in A \Rightarrow \tau_A = k$

Hitting times

We conclude that

$$h(i) = \sum_{j \in S} p(i,j)\, h(j) \qquad (**)$$

This gives a system of linear equations + boundary conditions

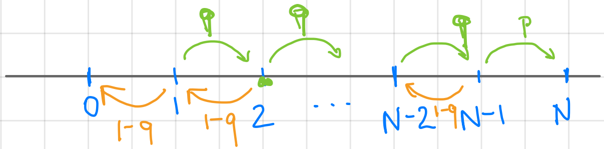
$$h(i) = \begin{cases} 1, & i \in A \\ 0, & i \in B \end{cases} \qquad (***)$$

If S is finite, denote $\bar{h} := (h(1), h(2), \dots, h(|S|))$. Then $(**)$ becomes $\bar{h} = P h$

Example 2.6 (X_n) random walk on $\{0,1,2,\dots,N\}$, not necessarily symmetric, $p(i,i+1) = q$, $p(i,i-1) = 1-q$, $q \in [0,1]$

Let $i \in \{1,2,\dots,N-1\}$. Compute

$\mathbb{P}[X_n \text{ reaches } N \text{ before } 0 \mid X_0 = i]$



Hitting times for random walks

Denote $A=\{N\}$, $B=\{0\}$. Need $\mathbb{P}[\tau_A<\tau_B \mid X_0=i]=h(i)$

- boundary conditions $h(0)=0$, $h(N)=1$

Consider $0<i<N$

- recall $p(i,j)=\begin{cases} q, & j=i+1 \\ 1-q, & j=i-1 \\ 0, & \text{otherwise} \end{cases}$, so $(**)$ becomes

$h(i)=\sum_{j\in S} p(i,j)\,h(j)$

$$h(i)=q\,h(i+1)+(1-q)\,h(i-1) \qquad \forall\, i\in\{1,\dots,N-1\}$$

$$(1-q)\big(h(i)-h(i-1)\big)=q\big(h(i+1)-h(i)\big)$$

- if $q=0$, then $h(i)=h(i-1)=h(0)=0$
- if $q=1$, then $h(i)=1$
- if $q\in(0,1)$, denote $\Delta h(i):=h(i)-h(i-1)$, $\theta:=\dfrac{1-q}{q}$

Hitting times for random walks

$$\Delta h(1) = \Delta h(1)$$

$$\oplus \begin{cases} \Delta h(2) = \theta\, \Delta h(1) \\ \Delta h(3) = \theta\, \Delta h(2) = \theta^2 \Delta h(1) \\ \vdots \\ \Delta h(N) = \theta^{N-1} \Delta h(1) \end{cases}$$

Take the sum of the first i equations

$$\text{LHS}: \quad \Delta h(1) + \Delta h(2) + \cdots + \Delta h(i) = \cancel{h(1)} - h(0) + h(2) - \cancel{h(1)} \cdots = h(i) - \underset{0}{\underset{\parallel}{h(0)}}$$

$$\text{RHS}: \quad (1 + \theta + \theta^2 + \cdots + \theta^{i-1})\, \Delta h(1)$$

$$\Rightarrow \forall\, i \in \{2, 3, \ldots, N\} \qquad h(i) = \sum_{\ell=0}^{i-1} \theta^\ell\, \Delta h(1)$$

$$h(N) = 1 = \sum_{\ell=0}^{N-1} \theta^\ell\, \Delta h(1) \quad \Rightarrow \quad \Delta h(1) = h(1) = \frac{1}{\sum_{\ell=0}^{N-1} \theta^\ell}$$

$$\Rightarrow \quad h(i) = \sum_{\ell=0}^{i-1} \theta^\ell \Big/ \sum_{\ell=0}^{N-1} \theta^\ell \qquad \text{for } i \in \{1, \ldots, N-1\}$$

Gambler's ruin

Suppose you have 100 \$, at each game you bet 1 \$, and you stop either when your fortune reaches 200\$ or when you lose everything. [$N=200$, $h(100)-?$]

(fair game) If probability of winning is 0.5 ($q=0.5$)

then $\theta = \frac{0.5}{0.5} = 1$, $h(100) = \frac{100}{200} = \frac{1}{2} = 0.5$

(real gambling) If probability of winning is $\frac{18}{38}$ ($q = 0.474$)

then $h(100) = \frac{1-\theta^{100}}{1-\theta^{200}} = 0.000027$