MATH 285: Stochastic Processes

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Today: Hitting times. First step analysis. Stopping times

- Homework 1 is due on Friday, January 14, 11:59 PM

Expected hitting times

Let $(X_n)_{n\geq 0}$ be a Markov chain with transition probabilities $p(i,j)$ and state space S.

Notation: $\mathbb{P}_i[Y] = \mathbb{P}[Y \mid X_0 = i]$, $\mathbb{E}_i[Y] = \mathbb{E}[Y \mid X_0 = i]$

Let $A \subset S$, $\tau_A := \min\{n \geq 0 : X_n \in A\}$

Q: How long (on average) does it take to reach A?

Compute $\mathbb{E}_i[\tau_A] = \mathbb{E}[\tau_A \mid X_0 = i] =: g(i)$

By definition, $\mathbb{E}_i[Y] = \sum_{k=1}^{\infty} k\, \mathbb{P}[\underline{Y=k} \mid X_0 = i]$ $\quad (Y \in \{0,1,2,\dots\})$

First step analysis (conditioning on the first step)

$$g(i) = \mathbb{E}_i[\tau_A] = \sum_{j\in S} \mathbb{E}[\tau_A \mid X_1 = j, X_0 = i]\, \mathbb{P}[X_1 = j \mid X_0 = i]$$

Expected hitting times

If $i \in A$, then $g(i) = 0$. Suppose $i \notin A$.

Then

$$\mathbb{P}[\tau_A = k \mid X_1 = j, X_0 = i] = \mathbb{P}[X_0 \notin A, X_1 \notin A, \dots, X_{k-1} \notin A, X_k \in A \mid X_1 = j, X_0 = i]$$

$$= \mathbb{P}[X_0 \notin A, X_1 \notin A, \dots, X_{k-2} \notin A, X_{k-1} \in A \mid X_0 = j]$$

$$= \mathbb{P}[\tau_A = k-1 \mid X_0 = j]$$

Compute the expectation

$$g(i) = \sum_{j \in S} \mathbb{E}[\tau_A \mid X_1 = j\ X_0 = i]\, \mathbb{P}[X_1 = j \mid X_0 = i]$$

$$= \sum_{j \in S} \sum_{k=1}^{\infty} k\, \mathbb{P}[\tau_A = k \mid X_1 = j, X_0 = i]\, p(i,j) = \sum_{j \in S} \sum_{k=1}^{\infty} k\, \mathbb{P}[\tau_A = \underbrace{k-1}_{=\ell} \mid X_0 = j]\, p(i,j)$$

$$= \underbrace{\sum_{j \in S} \sum_{\ell=0}^{\infty} \ell\, \mathbb{P}[\tau_A = \ell \mid X_0 = j]}\, p(i,j) + \underbrace{\sum_{j \in S} \sum_{\ell=0}^{\infty} \mathbb{P}[\tau_A = \ell \mid X_0 = j]\, p(i,j)}$$

$$= \sum_{j \in S} \mathbb{E}_j[\tau_A]\, p(i,j) + 1 = \sum_{j \in S} g(j)\, p(i,j) + 1$$

Expected hitting times

Conclusion:

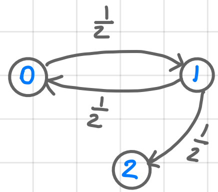
$$\begin{cases} g(i) = 1 + \sum_{j \in S} p(i,j)\, g(j) & \text{if } i \notin A \\ g(i) = 0 & \text{if } i \in A \end{cases}$$

Example 3.2 On average how many times do we need to toss a coin to get two consecutive heads?

Denote by X_n the number of consecutive heads after n^{th} toss.

$X_n \in \{0, 1, 2\}$, $\quad P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ 2 & 0 & 0 & 1 \end{array}$

$A = \{2\}$



$g(2) = 0 \qquad g(1) = 1 + \frac{1}{2} g(0) + \frac{1}{2} \underbrace{g(2)}^{=0} \qquad g(0) = \frac{3}{2} g(1)$

$g(0) = 1 + \frac{1}{2} g(0) + \frac{1}{2} g(1) \qquad g(1) = 4 \quad g(0) = 6$

Starting from state 0 it takes on average 6 tosses to reach state 2.

Stopping Times

Def 3.3 Let $(X_n)_{n\geq 0}$ be a discrete time stochastic process. A stopping time is a random variable $T \in \{0,1,2,\dots\} \cup \{\infty\}$ such that for each n the event $\{T=n\}$ depends only on $X_0, X_1, \dots, X_n$

Examples $T_1 = \min\{n \geq 0 : X_n = i\}$ is a stopping time

$$\{T_1 = n\} = \{X_0 \neq i, X_1 \neq i, \dots, X_{n-1} \neq i, X_n = i\}$$

$T_2 = \max\{n \geq 0 : X_n = i\}$ is not a stopping time

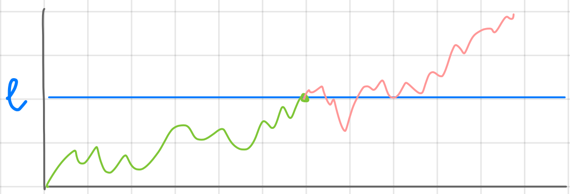
$$\{T_2 = n\} = \{X_n = i, X_{n+1} \neq i, \dots\}$$

Recall Markov property: If (X_n) is Markov(λ, P), then conditional on $X_m = \ell$, the process $(X_{m+n})_{n\in\mathbb{N}}$ is Markov(δ_ℓ, P) independent of $X_0, X_1, \dots, X_m$

Strong Markov property

Proposition 3.6 Let (X_n) be a time-homogeneous Markov chain with state space S and transition probabilities $p(i,j)$. Let T be a stopping time, $\ell \in S$ and $\mathbb{P}[X_T = \ell] > 0$. Then, conditional on $X_T = \ell$, $(X_{T+n})_{n\geq 0}$ is a time-homogeneous Markov chain with transition probabilities $p(i,j)$ independent of $X_0, \dots, X_T$. In other words, if A is an event that depends only on $X_0, X_1, \dots, X_T$ and $\mathbb{P}[A \cap \{X_T = \ell\}] > 0$ then for all $n \geq 0$ and all $i_0, i_1, \dots, i_n \in S$

$$\mathbb{P}[X_{T+1} = i_1, X_{T+2} = i_2, \dots, X_{T+n} = i_n \mid A \cap \{X_T = \ell\}] = p(\ell, i_1)\, p(i_1, i_2) \cdots p(i_{n-1}, i_n)$$



Proof. Use the partition $\{\{T=m\}\}_{m=0}^{\infty}$ and apply Markov property (see the notes)

Classification of states : recurrence and transience

Let (X_n) be a Markov chain with state space S.

Def 4.1 A state $i \in S$ is called recurrent if

$$\mathbb{P}_i[X_n = i \text{ for infinitely many } n] = 1$$

A state $i \in S$ is called transient if

$$\mathbb{P}_i[X_n = i \text{ for infinitely many } n] = 0$$

Remark

Let $T_{i,k}$ = time X_n (starting from i) visits state i k^{th} time

$T_{i,1} = 0$, $T_{i,k+1} = \min\{n > T_{i,k} : X_n = i\}$

Then, for $k \geq 2$, $T_{i,k}$ are stopping times. Indeed,

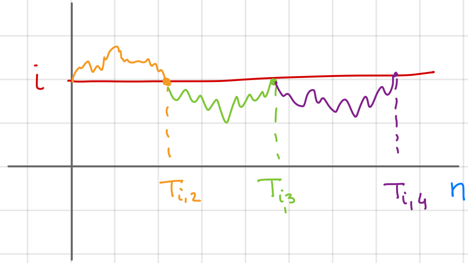
$$\{T_{i,2} = m\} = \{X_1 \neq i, X_2 \neq i, \dots, X_{m-1} \neq i, X_m = i\}$$

$$\{T_{i,k} = m\} = \bigsqcup_{\ell=k-2}^{m-1} \{T_{i,k-1} = \ell, T_{i,k} = m\} = \bigsqcup_{\ell=k-2}^{m-1} \{\underbrace{T_{i,k-1} = \ell}_{\text{depends on } X_0, \dots, X_\ell}, X_{\ell+1} \neq i, \dots, X_{m-1} \neq i, X_m = i\}$$

Classification of states : recurrence and transience

Denote $T_i := T_{i,2} = \min\{n>0 : X_n = i\}$ and

$$r_i := \mathbb{P}_i[T_i < \infty]$$



Theorem 4.2

Let $i \in S$. Then

(1) i is recurrent $\iff r_i = 1 \iff \sum_{n=0}^{\infty} p_n(i,i) = \infty$

(2) i is transient $\iff r_i < 1 \iff \sum_{n=0}^{\infty} p_n(i,i) < \infty$

Proof. Step 1: By the Strong Markov property

$$\mathbb{P}_i[T_{i,k+1} < \infty \mid T_{i,k} < \infty] = \mathbb{P}[T_{i,2} < \infty] = r_i$$

$$\mathbb{P}_i[T_{i,k+1} < \infty] = \mathbb{P}[T_{i,k+1} < \infty \mid T_{i,k} < \infty]\,\mathbb{P}[T_{i,k} < \infty] = \cdots = r_i^k$$

Step 2: Denote $N_i := \sum_{n=0}^{\infty} \mathbb{1}_{X_n = i}$ ← # times (X_n) visits state i

$\forall\ k \geq 1$, $\{N_i \geq k\} = \{T_{i,k} < \infty\}$, so $\mathbb{P}_i[N_i \geq k] = \mathbb{P}_i[T_{i,k} < \infty] = r_i^{k-1}$

Classification of states: recurrence and transience

Thus $\mathbb{E}_i[N_i] = \sum_{k=1}^{\infty} \mathbb{P}_i[N_i \geq k] = \sum_{k=1}^{\infty} r_i^{k-1} = \sum_{\ell=0}^{\infty} r_i^{\ell}$

$$\mathbb{E}_i[N_i] = \mathbb{E}_i\Big[\sum_{n=0}^{\infty} \mathbb{1}_{X_n=i}\Big] = \sum_{n=0}^{\infty} \mathbb{P}_i[X_n=i] = \sum_{n=0}^{\infty} p_n(i,i)$$

Since $r_i \in [0,1]$, $\sum_{\ell=0}^{\infty} r_i^{\ell} = \infty \iff r_i = 1$, $\sum_{\ell=0}^{\infty} r_i^{\ell} < \infty \iff r_i < 1$

Step 3: $r_i = 1 \iff \forall k \;\; \mathbb{P}_i[N_i \geq k] = 1$, i.e., i is recurrent

Step 4: $r_i < 1 \iff \mathbb{P}_i[N_i \geq k] = r_i^k \to 0,\ k \to \infty$,

so $\mathbb{P}_i[N_i = \infty] = 0$, i.e., i is transient

$$\sum_{n=0}^{\infty} p_n(i,i) = \sum_{\ell=0}^{\infty} r_i^{\ell} = \frac{1}{1-r_i}$$

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