MATH 285: Stochastic Processes

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Today: Irreducible Markov chains. Random walks on graphs

- Homework 1 is due on Friday, January 14, 11:59 PM

Classification of states : recurrence and transience

Let (X_n) be a Markov chain with state space S.

Def 4.1 A state $i \in S$ is called recurrent if

$$\mathbb{P}_i (X_n = i \text{ for infinitely many } n) = 1$$

A state $i \in S$ is called transient if

$$\mathbb{P}_i (X_n = i \text{ for infinitely many } n) = 0$$

Denote $T_i := T_{i,2} = \min\{n > 0 : X_n = i\}$ and $r_i := \mathbb{P}_i [T_i < \infty]$

Theorem 4.2

Let $i \in S$. Then

(1) i is recurrent $\iff r_i = 1 \iff \sum_{n=0}^{\infty} p_n(i,i) = \infty$

(2) i is transient $\iff r_i < 1 \iff \sum_{n=0}^{\infty} p_n(i,i) < \infty$.

Recurrence and transience of RW

Example 4.5

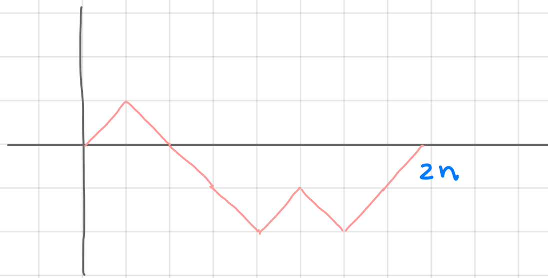
Let (X_n) be a random walk on $\mathbb{Z}$, $p(i,j) = \begin{cases} p, & j = i+1 \\ 1-p, & j = i-1 \\ 0, & \text{otherwise} \end{cases}$

Fix $i \in \mathbb{Z}$. Is i recurrent or transient?

Use the $\sum_{n=0}^{\infty} p_n(i,i)$ criterion.

Notice that $p_n(i,i) = 0$ if n is odd

Goal: compute $\sum_{n=0}^{\infty} p_{2n}(i,i)$



$p_{2n}(i,i) =$ (trivial for $p=0$ or $p=1$)

Case 1: $p \in (0,1)$, $p \neq \frac{1}{2}$. Then $p(1-p) < \frac{1}{4}$

$$\sum_{n=0}^{\infty} p_{2n}(i,i) = \sum_{n=0}^{\infty} \binom{2n}{n} \left(p(1-p)\right)^n$$

$\binom{2n}{n} < 4^n$ $\Rightarrow$ all states are

Recurrence and transience of RW

Case 2: $p = \frac{1}{2}$

$$\binom{2n}{n} = \frac{(2n)!}{n!\,n!} \leftarrow \text{use Stierling's approximation}$$

$$n! \sim \sqrt{2\pi n}\,\frac{n^n}{e^n}$$

$$\binom{2n}{n} \sim$$

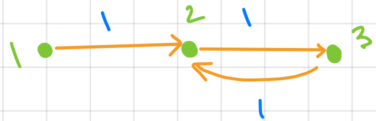
$$\sum_{n=0}^{\infty} p_n(i,i) =$$

$\Rightarrow$ all states are

Irreducibility

Is it always true that either all states are recurrent or all states are transient?

Example



transient

recurrent

Def 4.7 Markov chain is called irreducible if for any $i, j \in S$ there exists $n \in \mathbb{N}$ s.t.

Prop. 4.8 If (X_n) is irreducible, then either all states are recurrent or all states are transient.

Proof. Suppose i is transient, $j \in S$, $p_{n_0}(i,j) > 0$, $p_{n_1}(j,i) > 0$

Then $\forall\, m \in \mathbb{N}$ $\quad p_{n_0+m+n_1}(i,i) \geq$

$$\sum_{m=0}^{\infty} p_m(j,j) \leq$$

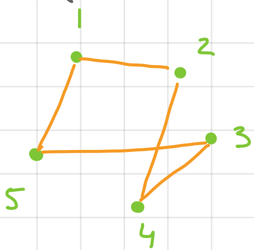
Graphs

Def 5.1 A graph $G=(V,E)$ is a collection of vertices V and relations E on $V\times V$ (which we call edges).

For $x,y\in V$ we write $x\sim y$ to mean $(x,y)\in E$.

E is assumed to be anti-reflexive ($x\not\sim x$, no loops) and symmetric (if $x\sim y$ then $y\sim x$, indirected graph).

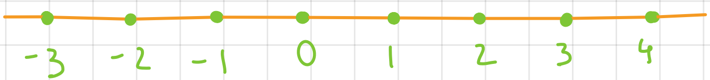
Example



$V=$

$E=$

Example



$V=$

$E=$

Valence of a vertex $x\in V$: $v_x=$

Simple random walks of graphs

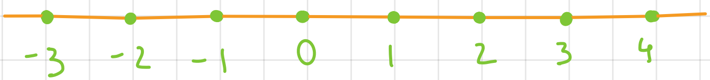
Def. 5.2 The simple random walk on the graph $G=(V,E)$ is the Markov chain (X_n) with state space V and transition probabilities $p(i,j)$ s.t.

and (X_n) is called symmetric

if for all j s.t. $i \sim j$.

Example 5.3 RW on $\mathbb{Z}$

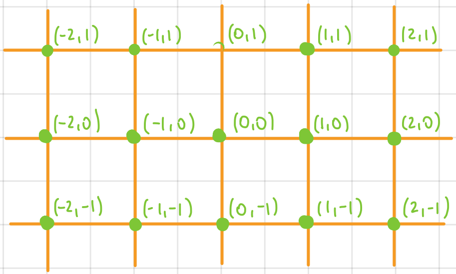
$$p(i,j) = \begin{cases} \frac{1}{2}, & j = i+1 \\ \frac{1}{2}, & j = i-1 \\ 0, & \text{otherwise} \end{cases}$$



Example 5.4.

RW on $\mathbb{Z}^d$

$$\|x\|_1 = \sum_{m=1}^{d} |x_m|$$



$$V = \mathbb{Z}^d = \{ \underset{\bar{i}}{\underbrace{(i_1, \dots, i_d)}} : i_m \in \mathbb{Z} \}$$

$\bar{i} \sim \bar{j}$ iff $\|\bar{i} - \bar{j}\|_1 = 1$

$$\Rightarrow \quad v_{\bar{i}} = \frac{1}{2d}$$

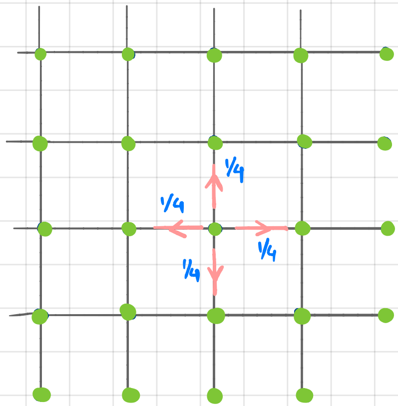
SRW on $\mathbb{Z}^d$

Remark For any $d \in \mathbb{N}$, simple random walks on $\mathbb{Z}^d$ are irreducible $\Rightarrow$ all states are in the same class

SSRW on $\mathbb{Z}^d$, $d \in \{1, 2, 3\}$

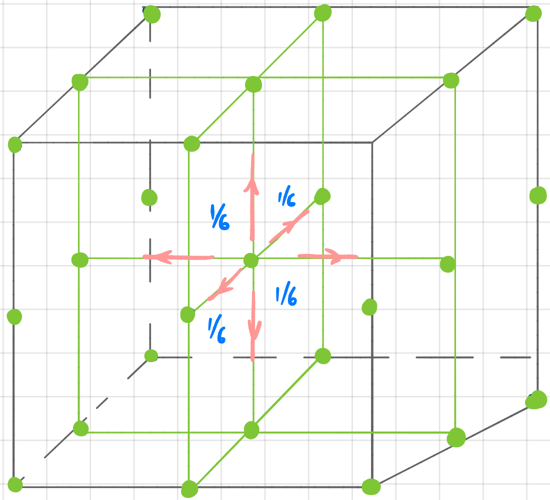
transient

recurrent



transient

recurrent



transient

recurrent

Simple symmetric RW on $\mathbb{Z}^3$

As for $d=1$, $p_n(i,i) = 0$ if n is odd

Goal: determine if $\sum_{n=0}^{\infty} p_{2n}(i,i)$ is finite or not.

Take $i = \bar{0} = (0,0,0)$ for simplicity.

$$p_{2n}(\bar{0},\bar{0}) =$$

i steps $(+1,0,0)$

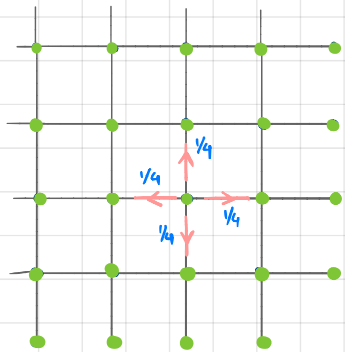
i steps $(-1,0,0)$

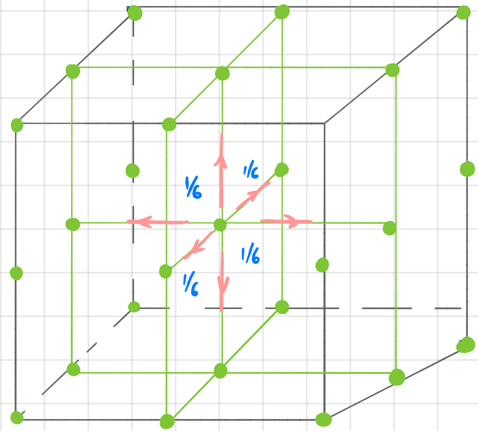
j steps $(0,+1,0)$

j steps $(0,-1,0)$

k steps $(0,0,+1)$

k steps $(0,0,-1)$





Simple symmetric RW on $\mathbb{Z}^3$

Step 1: $\#\{$ paths from $\bar{0}$ to $\bar{0}$ of length $2n\}$ $=$

$$p_{2n}(\bar{0},\bar{0}) = \sum_{\substack{i,j,k\geq 0 \\ i+j+k=n}} \frac{(2n)!}{(i!\, j!\, k!)^2} \cdot \left(\frac{1}{6}\right)^{2n} =$$

Step 2:

$$\sum_{\substack{i,j,k\geq 0 \\ i+j+k=n}} \binom{n}{i,j,k} = \qquad , \text{ so } \quad \sum_{\substack{i,j,k\geq 0 \\ i+j+k=n}} \binom{n}{i,j,k}\left(\frac{1}{3}\right)^{n} =$$

Step 3: If $a_i \geq 0$ and $a_i \leq M$, then

and thus

$$\sum_{\substack{i,j,k\geq 0 \\ i+j+k=n}} \binom{n}{i,j,k}^2 \left(\frac{1}{3}\right)^{2n} \leq$$

Simple symmetric RW on $\mathbb{Z}^3$

Steps 1-3 imply that

$$P_{2n}(\bar{0},\bar{0}) \leq \binom{2n}{n}\left(\frac{1}{2}\right)^{2n} \max_{\substack{i,j,k\geq 0 \\ i+j+k=n}} \binom{n}{i,j,k}\left(\frac{1}{3}\right)^{n} \qquad (*)$$

Step 4: If $n = 3m$, then $\max\limits_{\substack{i,j,k\geq 0 \\ i+j+k=n}} \binom{n}{i,j,k} =$

Step 5: $\dfrac{(3m)!}{m!\,m!\,m!}\left(\dfrac{1}{3}\right)^{3m} \sim$

Steps 4-5 + $(*)$ + asymptotics for $\binom{2n}{n}\left(\frac{1}{2}\right)^{2n} \sim \frac{1}{\sqrt{\pi n}}$ gives

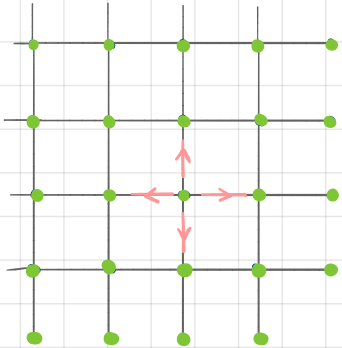
$$P_{6m}(\bar{0},\bar{0}) \sim \frac{1}{\sqrt{\pi n}} \cdot \frac{\sqrt{2\pi n}}{(2\pi m)^{3/2}} =$$

and

Simple symmetric RW on $\mathbb{Z}^3$

Step 6: $\quad p_{6m}(\bar{0},\bar{0}) \geq \qquad \forall\, m \in \mathbb{N}$

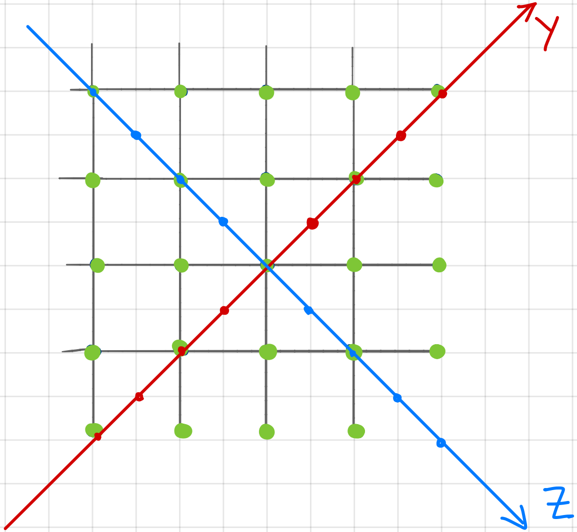
$\qquad\quad p_{6m}(\bar{0},\bar{0}) \geq \qquad \forall\, m \in \mathbb{N}$



Conclusion: $\displaystyle \sum_{n=0}^{\infty} p_{2n}(\bar{0},\bar{0}) \leq \left(1+6^2+6^4\right) \sum_{m=0}^{\infty} p_{6m}(\bar{0},\bar{0}) < \infty$

All states of a SSRW on $\mathbb{Z}^3$ are

Simple symmetric random walk on $\mathbb{Z}^2$



Y_n = projection of X_n on $y=x$

Z_n = projection of X_n on $y=-x$

$X_n = (i,j) \Leftrightarrow Y_n = i+j\,,\ Z_n = j-i$

$X_n = (0,0) \Leftrightarrow Y_n = 0,\ Z_n = 0$

Let (Y_n) and (Z_n) be two independent SSRW on $\mathbb{Z}$

Define $\tilde{X}_n =$

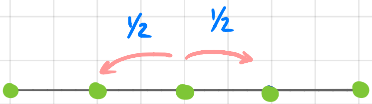
Then $(\tilde{X}_n)$ is a ; $\tilde{X}_n = (0,0) \Leftrightarrow$

$p_n^{\tilde{x}}(\bar{0},\bar{0}) =$

$= p_n^{Y}(0,0)\, p_n^{Z}(0,0) \sim$

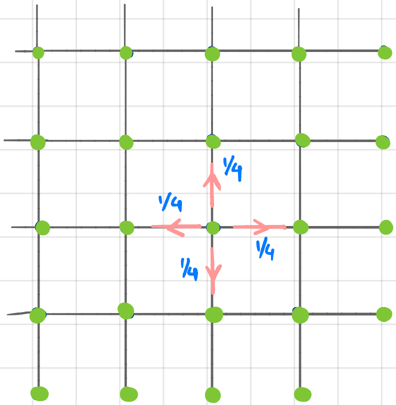
$\Rightarrow \sum_{n=0}^{\infty} p_n^{\tilde{x}}(\bar{0},\bar{0}) \sim$

SSRW on $\mathbb{Z}^d$, $d \in \{1, 2, 3\}$



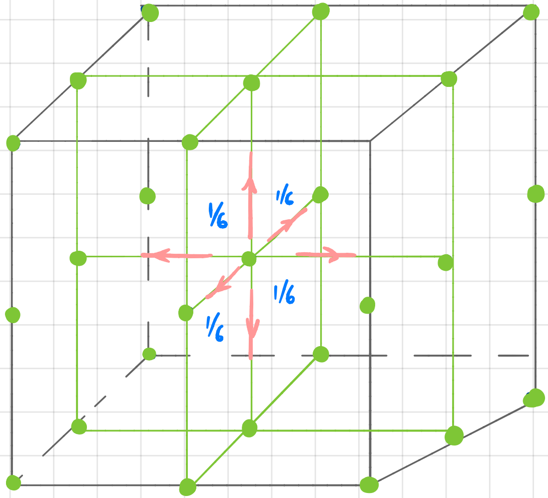
transient

recurrent



transient

recurrent



transient

recurrent