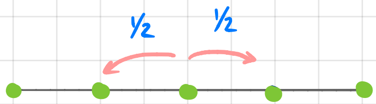
MATH 285: Stochastic Processes

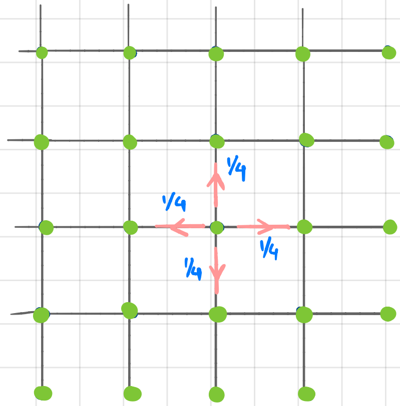
math-old.ucsd.edu/~ynemish/teaching/285

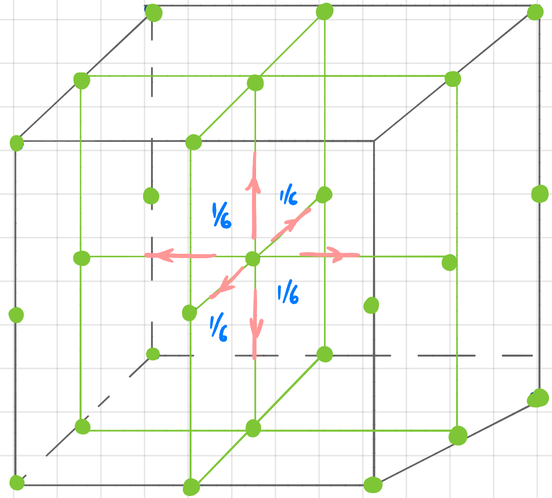
Today: Stationary distribution

- Homework 1 is due on Friday, January 14, 11:59 PM

SSRW on $\mathbb{Z}^d$, $d \in \{1,2,3\}$







~~transient~~

recurrent

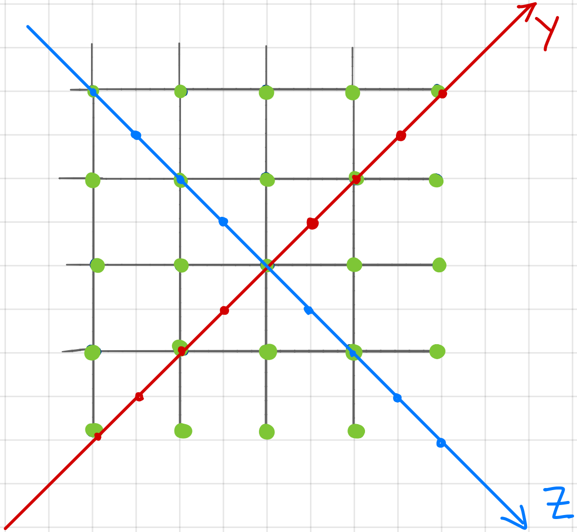
transient

recurrent

transient

~~recurrent~~

Simple symmetric random walk on $\mathbb{Z}^2$



Y_n = projection of X_n on $y=x$

Z_n = projection of X_n on $y=-x$

$X_n=(i,j) \Leftrightarrow Y_n = i+j\ ,\ Z_n = j-i$

$X_n=(0,0) \Leftrightarrow Y_n=0,\ Z_n=0$

Let (Y_n) and (Z_n) be two independent SSRW on $\mathbb{Z}$

Define $\tilde{X}_n = \left(\dfrac{Y_n+Z_n}{2}\ ,\ \dfrac{Y_n-Z_n}{2} \right)$

Then $(\tilde{X}_n)$ is a SSRW on $\mathbb{Z}^2$; $\tilde{X}_n = (\bar{0},\bar{0}) \Leftrightarrow Y_n=0\ ,\ Z_n=0$

$$p_n^{\tilde{X}}(\bar{0},\bar{0}) = \mathbb{P}[\tilde{X}_n=(0,0) \mid \tilde{X}_0=(0,0)] = \mathbb{P}[Y_n=0,\ Z_n=0 \mid Y_0=0,\ Z_n=0]$$

$$= p_n^Y(0,0)\, p_n^Z(0,0) \sim \left(\frac{1}{\sqrt{\pi n}}\right)^2 = \frac{1}{\pi n} \Rightarrow \sum_{n=0}^{\infty} p_n^{\tilde{X}}(\bar{0},\bar{0}) \sim \sum_{n=0}^{\infty} \frac{1}{\pi n} = \infty$$

Markov processes

Let (X_n) be a Markov chain with initial distribution λ and transition matrix P.

- Distribution of X_n : λP^n
- First step analysis:
  - absorption probabilities (gambler's ruin)
  - mean hitting times (two consecutive heads)
- Class structure : recurrence / transience
  - criteria
  - SSRW on $\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^3$
- Irreducibility

Long-run behavior of Markov chains

Denote by π_n the distribution of X_n, i.e.,

$$\pi_n = (\mathbb{P}[X_n = 1], \mathbb{P}[X_n = 2], \dots, \mathbb{P}[X_n = |S|])$$

$\pi_n = \pi_0 P^n$ (follows from the Chapman-Kolmogorov eqs.)

What happens with π_n as $n \to \infty$?

P^n as $n \to \infty$ for a stochastic matrix P

Examples:

① $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad P_1^n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\pi_n = \pi_0$

② $P_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad P_2^{2n} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad P_2^{2n+1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\pi_{2n} = \pi_0$, $\pi_{2n+1} = (\pi_0(2), \pi_0(1))$

③ $P^n = \begin{bmatrix} 0.04 & 0.05 & 0.91 \\ 0.04 & 0.05 & 0.91 \\ 0.04 & 0.05 & 0.91 \end{bmatrix}$

$\pi_n = \pi_0 P_n$

$\pi_n = (\pi_0(1)\,0.04 + \pi_0(2)\cdot 0.04 + \pi_0(3)\,0.04$
$= 0.04,\ 0.05,\ 0.91)$

Stationary distribution

Def 6.1 Let $(X_n)_{n\geq 0}$ be a Markov chain with state space S and transition matrix P. A vector $\pi = (\pi(i))_{i\in S}$ is called a stationary distribution if

$\pi(i) \geq 0$ for all $i \in S$, $\sum_{i\in S} \pi(i) = 1$ and

$\pi P = \pi$ (*)

If π is the stationary distribution and $\pi_0 = \pi$, then

$\forall n$ $\pi_n = \pi P^n = \pi P \cdot P^{n-1} = \pi \cdot P^{n-1} = \dots = \pi$

In order to find the stationary distribution we have to solve the linear system (*):

- π is the left eigenvector of P with e.v. 1

Stationary distribution

Q 1 : Existence of the stationary distribution

Q 2 : Uniqueness of the stationary distribution

Q 3 : Convergence to the stationary distribution

Examples 6.3. (1) $S = \mathbb{Z}$, $p(i, i+1) = 1 \quad \forall i \in \mathbb{Z}$ (deterministic).

Then $\forall i \quad \lim_{n\to\infty} \mathbb{P}[X_n = i] = 0$, so st. distr. does not exist

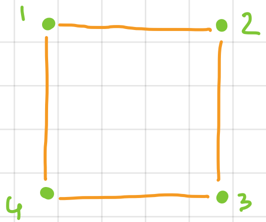
(2)

$S = \{1,2,3,4\}$, $P = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}$.

Then $\pi = (\frac{1}{2}, \frac{1}{2}, 0, 0)$ and $\pi' = (0, 0, \frac{1}{2}, \frac{1}{2})$ are both stationary distributions.

(3)

SSRW on



If $X_0 = 1$, then

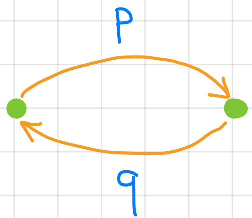
$\mathbb{P}[X_{2n+1} \in \{1,3\}] = 0$

$\mathbb{P}[X_{2n} \in \{1,3\}] = 1$

$P = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix}$

$\pi = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$

General 2-state Markov chain



$$P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} \qquad p, q \in [0,1]$$

$$\det(P - \lambda I) = (1-p-\lambda)(1-q-\lambda) - pq = \lambda^2 + \lambda(p+q-2) + 1-p-q = 0$$

$\hookrightarrow$ eigenvalues are $1,\ 1-p-q$

$P - \lambda I$

$$P - I = \begin{pmatrix} -p & p \\ q & -q \end{pmatrix} \qquad \begin{cases} -\pi(1)\,p + \pi(2)\cdot q = 0 \\ \pi(1) + \pi(2) = 1 \end{cases} \qquad P - (1-p-q)I = \begin{bmatrix} q & p \\ q & p \end{bmatrix}$$

Case 1: $p, q \in \{0, 1\}$

$$P \in \left\{ \underset{p=0,\ q=1}{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}},\ \underset{p=1,\ q=0}{\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}},\ \underset{p=0,\ q=0}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}},\ \underset{p=1,\ q=1}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \right\}$$

$\pi = (1, 0)$	$\pi = (0, 1)$	any π, $\pi(1) + \pi(2) = 1$	$\pi = (\frac{1}{2}, \frac{1}{2})$
unique	unique	not unique	unique
$P^n = P$	$P^n = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$	$P^n = P$	$P^{2n+1} = P$, $P^{2n} = I$ no convergense

General 2-state Markov chain

Case 2: $p \in (0,1)$ or $q \in (0,1)$

$$\begin{cases} -\pi(1)p + \pi(2)q = 0 \\ \pi(1) + \pi(2) = 1 \end{cases} \Rightarrow \pi(1) = \frac{q}{p+q}, \quad \pi(2) = \frac{p}{p+q}$$

$$(x, y)\begin{pmatrix} q & p \\ q & p \end{pmatrix} = (0,0) \Rightarrow x = -y \qquad (x,y) = (1,-1)$$

$$Q^{-1} = \begin{pmatrix} \frac{q}{p+q} & \frac{p}{p+q} \\ 1 & -1 \end{pmatrix}, \quad Q = -\begin{bmatrix} -1 & -\frac{p}{p+q} \\ -1 & \frac{q}{p+q} \end{bmatrix} = \begin{bmatrix} 1 & \frac{p}{p+q} \\ 1 & -\frac{q}{p+q} \end{bmatrix}$$

$$P = Q\begin{bmatrix} 1 & 0 \\ 0 & 1-p-q \end{bmatrix}Q^{-1} \Rightarrow P^n = Q\begin{bmatrix} 1 & \\ & (1-p-q)^n \end{bmatrix}Q^{-1} \xrightarrow[n\to\infty]{} Q\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}Q^{-1}$$

$$\lim_{n\to\infty} P^n = \begin{bmatrix} 1 & \frac{p}{p+q} \\ 1 & -\frac{q}{p+q} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{q}{p+q} & \frac{p}{p+q} \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} \frac{q}{p+q} & \frac{p}{p+q} \end{bmatrix} = \begin{bmatrix} \pi \\ \pi \end{bmatrix}$$

$\lim_{n\to\infty} \pi_n = \pi$ regardless of initial distribution.

General Markov chain with finite state space

Let (X_n) be a MC with finite state space S.

Suppose that $\pi = P\pi$, $P = QDQ^{-1}$ such that

$$Q = \begin{bmatrix} 1 & \\ 1 & * \\ \vdots & \\ 1 & \end{bmatrix} \quad Q^{-1} = \begin{bmatrix} \pi \\ \\ * \\ \\ \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & M & \\ 0 & & & \end{bmatrix}, \quad \lim_{n\to\infty} M^n = 0 .$$

Then $$\lim_{n\to\infty} P^n = \lim_{n\to\infty} Q D^n Q^{-1} = \begin{bmatrix} 1 & \\ \vdots & * \\ 1 & \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & O & \\ 0 & & & \end{bmatrix} \begin{bmatrix} \pi \\ * \\ \\ \end{bmatrix} = \begin{bmatrix} \pi \\ \pi \\ \vdots \\ \pi \end{bmatrix}$$