MATH 285: Stochastic Processes

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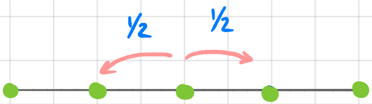
Today: Stationary distribution

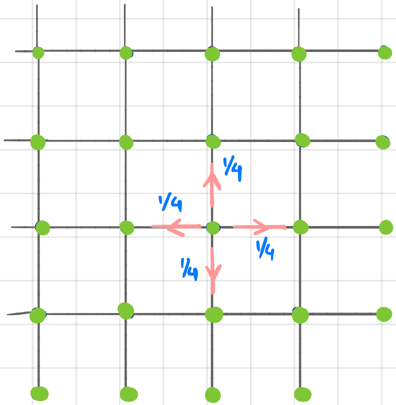
- Homework 1 is due on Friday, January 14, 11:59 PM

SSRW on $\mathbb{Z}^d$, $d \in \{1,2,3\}$



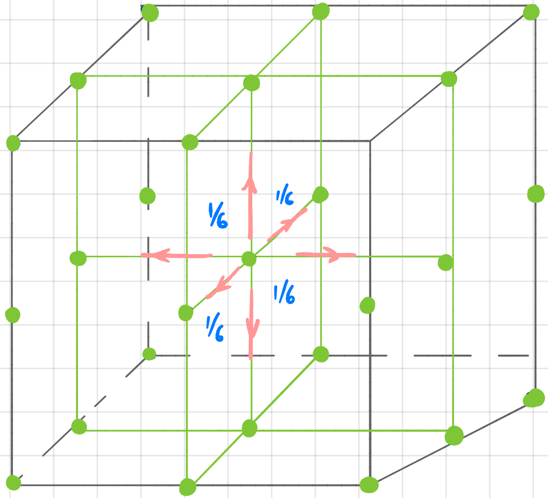
transient

recurrent



transient

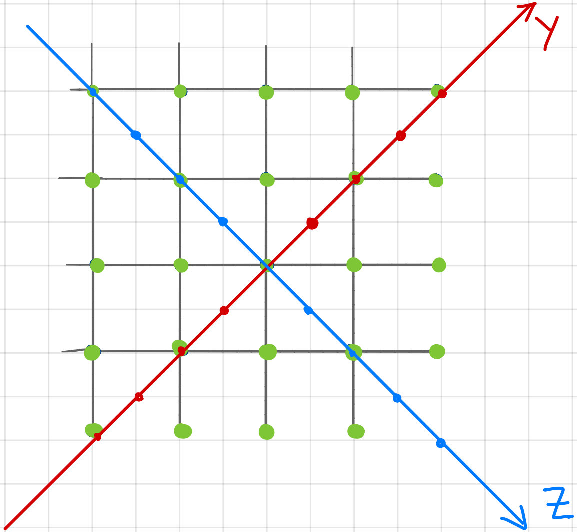
recurrent



transient

recurrent

Simple symmetric random walk on $\mathbb{Z}^2$



Y_n = projection of X_n on $y=x$

Z_n = projection of X_n on $y=-x$

$X_n=(i,j) \Leftrightarrow Y_n = i+j \,,\; Z_n = j-i$

$X_n=(0,0) \Leftrightarrow Y_n=0, \; Z_n=0$

Let (Y_n) and (Z_n) be two independent SSRW on $\mathbb{Z}$

Define $\tilde{X}_n =$

Then $(\tilde{X}_n)$ is a ; $\tilde{X}_n = (0,0) \Leftrightarrow$

$p_n^{\tilde{X}}(\bar{0},\bar{0}) =$

$= p_n^{Y}(0,0)\, p_n^{Z}(0,0) \sim$ $\quad \Rightarrow \sum_{n=0}^{\infty} p_n^{\tilde{X}}(\bar{0},\bar{0}) \sim$

Markov processes

Let (X_n) be a Markov chain with initial distribution λ and transition matrix P.

- Distribution of X_n : λP^n
- First step analysis:
  - absorption probabilities (gambler's ruin)
  - mean hitting times (two consecutive heads)
- Class structure : recurrence / transience
  - criteria
  - SSRW on $\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^3$
- Irreducibility

Long-run behavior of Markov chains

Denote by π_n the distribution of X_n, i.e.,

$$\pi_n = \left(\mathbb{P}[X_n = 1], \mathbb{P}[X_n = 2], \dots, \mathbb{P}[X_n = |S|] \right)$$

$\pi_n = \pi_0 P^n$ (follows from the Chapman-Kolmogorov eqs.)

What happens with π_n as $n \to \infty$?

for a stochastic matrix P

Examples:

① $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$P_1^n = \begin{bmatrix} \quad & \quad \end{bmatrix}$

$\pi_n =$

② $P_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$P_2^{2n} = \begin{bmatrix} \quad & \quad \end{bmatrix}$

$P_2^{2n+1} = \begin{bmatrix} \quad & \quad \end{bmatrix}$

$\pi_{2n+1} =$, $\pi_{2n} =$

③ $P^n = \begin{bmatrix} \quad & \quad \end{bmatrix}$

$\pi_n = \pi_0 P_n$

$=$

Stationary distribution

Def 6.1 Let $(X_n)_{n\geq 0}$ be a Markov chain with state space S and transition matrix P. A vector $\pi = (\pi(i))_{i\in S}$ is called a stationary distribution if

for all $i \in S$, and

(*)

If π is the stationary distribution and $\pi_0 = \pi$, then

$\forall n$

In order to find the stationary distribution we have to solve the linear system (*):

- π is the left eigenvector of P with e.v. 1

Stationary distribution

Q 1 : Existence of the stationary distribution

Q 2 : Uniqueness of the stationary distribution

Q 3 : Convergence to the stationary distribution

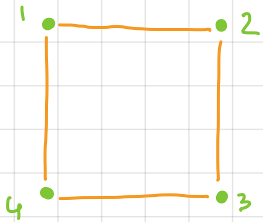
Examples 6.3. (1) $S = \mathbb{Z}$, $p(i, i+1) = 1 \ \forall i \in \mathbb{Z}$ (deterministic).

Then $\forall i$, so st. distr.

(2)

$S = \{1,2,3,4\}$, $P = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}$. Then $\pi =$ and $\pi' =$ are both stationary distributions.

(3)

SSRW on 

If $X_0 = 1$, then

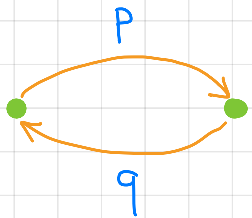
$\mathbb{P}[X_{2n+1} \in \{1,3\}] = 0$

$\mathbb{P}[X_{2n} \in \{1,3\}] = 1$

$P = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix}$

$\pi =$

General 2-state Markov chain



$$P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} \qquad p,q \in [0,1]$$

$$\det(P-\lambda I) = (1-p-\lambda)(1-q-\lambda) - pq = \lambda^2 + \lambda(p+q-2) + 1-p-q = 0$$

↳ eigenvalues

$$P - I = \begin{pmatrix} -p & p \\ q & -q \end{pmatrix}$$

$$P - (1-p-q)I =$$

Case 1:

$p,q \in \{0,1\}$

$$P \in \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

$p=0, q=1$	$p=1, q=0$	$p=0, q=0$	$p=1, q=1$
$\pi =$	$\pi =$		$\pi =$
$P^n =$	$P^n =$	$P^n =$	$P^{2n+1} =$, $P^{2n} =$

General 2-state Markov chain

Case 2: $p \in (0,1)$ or $q \in (0,1)$

$$\begin{cases} -\pi(1)p + \pi(2)q = 0 \\ \pi(1) + \pi(2) = 1 \end{cases} \Rightarrow$$

$$(x, y)\begin{pmatrix} q & p \\ q & p \end{pmatrix} = (0, 0) \Rightarrow$$

$$Q^{-1} = \begin{pmatrix} \frac{q}{p+q} & \frac{p}{p+q} \\ 1 & -1 \end{pmatrix}, \quad Q = \begin{bmatrix} \quad \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

$$P = \begin{bmatrix} \quad \end{bmatrix} \Rightarrow P^n = \begin{bmatrix} \quad \end{bmatrix}$$

$$\lim_{n\to\infty} P^n = \begin{bmatrix} 1 & \frac{p}{p+q} \\ 1 & -\frac{q}{p+q} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{q}{p+q} & \frac{p}{p+q} \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}\begin{bmatrix} \quad \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

$\lim_{n\to\infty} \pi_n = \pi$ regardless of initial distribution.

General Markov chain with finite state space

Let (X_n) be a MC with finite state space S.

Suppose that $\pi = P\pi$, $P = QDQ^{-1}$ such that

$$Q = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \quad Q^{-1} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \quad D = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix},$$

Then $$\lim_{n\to\infty} P^n = \lim_{n\to\infty} Q D^n Q^{-1} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

Enough to have the following: (use Jordan normal form)

1) 1 is a simple eigenvalue (1 is always an eigenvalue since $(P\mathbb{1})_i = \sum_j p(i,j) = 1$, so $P\mathbb{1} = \mathbb{1}$, $\mathbb{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ is an e.$\vec{v}$.)

2) There is a left eigenvector of 1 with all nonnegative entries

3) If λ is an eigenvalue of P and $\lambda = 1$, then $|\lambda| < 1$

Perron - Frobenius theorem

Theorem 6.5 Let M be an $N \times N$ matrix all of whose entries are strictly positive. Then

Moreover,

eigenspace contains a vector with

. Finally,

Let P be a stochastic matrix with all strictly positive entries. Then , therefore 1 is the PF eigenvalue:

with (left) eigenvector π with . If (X_n) is a MC with transition matrix P, then

(Enough to have P s.t. has strictly positive entries for some)