MATH 285: Stochastic Processes

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Today: Periodic, aperiodic, reducible, irreducible Markov chains

- Homework 2 is due on Friday, January 21 11:59 PM

First step analysis

Let (X_n) be a MC with state space S and transition matrix P.

hitting/absorption tim

Let $A \subset S$, $\tau_A = \min\{n \geq 0 : X_n \in A\}$, and denote

$h^A(i) := \mathbb{P}_i[\tau_A < \infty]$ (as in lecture 3 with $B = \emptyset$, so that $\tau_B = \infty$)

hitting/absorption probability

Then (lecture 2) $h^A(i)$ satisfies the system of linear equations

$$(*) \quad \begin{cases} h^A(i) = 1 & \text{if } i \in A \\ h^A(i) = \sum_{j \in S} p(i,j)\, h^A(j) & \text{if } i \notin A \end{cases}$$

The solution may be not unique.

Theorem 7.0 The vector of hitting probabilities $(h^A(i), i \in S)$ is the minimal nonnegative solution to $(*)$

(Minimal: if $(x(i), i \in S)$ satisfies $(*)$ and $x(i) \geq 0$ $\forall i$, then $x(i) \geq h^A(i)$ $\forall i$)

First step analysis

Proof of minimality: Let $(x(i), i \in S)$ be a nonnegative solution to $(*)$. Then $x(i) = 1$ for all $i \in A$ (so $x(i) \geq h^A(i)$)

For all $i \notin A$

$$x(i) = \sum_{j \in S} p(i,j) x_j = \sum_{j \in A} p(i,j) + \sum_{j \notin A} p(i,j) x(j)$$

$$= \sum_{j \in A} p(i,j) + \sum_{j \notin A} p(i,j) \Big(\sum_{k \in A} p(j,k) + \sum_{k \notin A} p(j,k) x(k) \Big)$$

$$= \sum_{j \in A} p(i,j) + \sum_{j \notin A} \sum_{k \in A} p(i,j) p(j,k) + \sum_{j \notin A} \sum_{k \notin A} p(i,j) p(j,k) x(k)$$

$$= \mathbb{P}_i [X_1 \in A] + \mathbb{P}_i [X_1 \notin A, X_2 \in A] + H_2^{\geq 0} = \cdots =$$

$$= \mathbb{P}_i [X_1 \in A] + \mathbb{P}_i [X_1 \notin A, X_2 \in A] + \cdots + \mathbb{P}_i [X_1 \notin A, \dots, X_{n-1} \notin A, X_n \in A] + H_n^{\geq 0}$$

$$= \mathbb{P}_i [\tau_A = 1] + \mathbb{P}_i [\tau_A = 2] + \mathbb{P}_i [\tau_A = n] + H_n^{\geq 0}$$

$$\Rightarrow \forall i \notin A \;\; \forall n \;\; x(i) \geq \mathbb{P}_i [\tau_A \leq n] \Rightarrow \forall i \notin A \;\; x(i) \geq \lim_{n \to \infty} \mathbb{P}_i [\tau_A \leq n] = h^A(i) \;\; \blacksquare$$

First step analysis

Denote $g^A(i) := \mathbb{E}_i[\tau_A]$ (mean hitting/absorption time)

Theorem 7.0' The vector of mean hitting times $(g^A(i), i \in S)$ is the minimal nonnegative solution to the system of linear equations

$$\begin{cases} g(i) = 1 + \sum_{j \in S} p(i,j)\, g(j) & \text{if } i \notin A \\ g(i) = 0 & \text{if } i \in A \end{cases}$$

Proof: Exercise.

Stationary distribution

Stationary distribution

$$\pi = \pi P$$

Q 1: Existence of the stationary distribution

Q 2: Uniqueness of the stationary distribution

Q 3: Convergence to the stationary distribution

General Markov chain with finite state space

Let (X_n) be a MC with finite state space S.

Suppose that $\pi = P\pi$, $P = QDQ^{-1}$ such that

$$Q = \begin{bmatrix} 1 & & \\ 1 & & \\ \vdots & * & \\ 1 & & \end{bmatrix} \qquad Q^{-1} = \begin{bmatrix} & \pi & \\ & * & \\ & & \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & M & \\ 0 & & & \end{bmatrix}, \quad \lim_{n\to\infty} M^n = 0 \ (**)$$

Then
$$\lim_{n\to\infty} P^n = \lim_{n\to\infty} Q D^n Q^{-1} = \begin{bmatrix} 1 & \\ 1 & * \\ \vdots & \\ 1 & \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{bmatrix} \begin{bmatrix} \pi \\ * \\ \end{bmatrix} = \begin{bmatrix} \pi \\ \pi \\ \vdots \\ \pi \end{bmatrix}$$

Enough to have the following: (use Jordan normal form)

1) 1 is a simple eigenvalue (1 is always an eigenvalue since $(P\mathbb{1})_i = \sum_j p(i,j) = 1$, so $P\mathbb{1} = \mathbb{1}$, $\mathbb{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ is an e. $\vec{v}$.)

2) There is a left eigenvector of 1 with all nonnegative entries

3) If λ is an eigenvalue of P and $\lambda \neq 1$, then $|\lambda| < 1$

Perron - Frobenius theorem

<u>Theorem 6.5</u> Let M be an $N\times N$ matrix all of whose entries are strictly positive. Then there is an eigenvalue $r>0$ such that all other eigenvalues λ satisfiy $|\lambda|<r$. Moreover, r is a simple eigenvalue and its one-dimensional eigenspace contains a vector with all strictly positive entries. Finally, r satisfies the bound $\min_i \sum_j M_{ij} \leq r \leq \max_i \sum_j M_{ij}$

Proof. No proof ■

Let P be a stochastic matrix with all strictly positive entries. Then $\sum_j P_{ij} = 1$, therefore 1 is the PF eigenvalue: simple with (left) eigenvector π with all positive entries. If (X_n) is a MC with transition matrix P, then $\lim_{n\to\infty} \pi_0 P^n = \pi$.

Perron - Frobenius Theorem

Enough if $\exists\, n_0 > 0$ s.t. all entries of P^{n_0} are strictly positive.

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Corollary 6.6 Let P be a stochastic matrix s.t. there exists $n_0 \in \mathbb{N}$ for which $\forall\, i,j \ [P^{n_0}]_{ij} > 0$

Then there exists a unique stationary distribution $\pi = \pi P$ and $\lim_{n\to\infty} \nu P^n = \pi$ for any distribution ν.

Proof. Use the fact that if $\vec{v} P = \lambda \vec{v}$, then $\vec{v} P^2 = \lambda \vec{v} P = \lambda^2 \vec{v}$ so P^n has the same eigenvectors as P, and eigenvalues are n-th powers of eigenvalues of P. By PF thm, 1 is ev of P^{n_0} with e$\vec{v}$s π and $\mathbb{1}$, therefore 1 is e.v. of P with e$\vec{v}$ π and $\mathbb{1}$

If λ is ev of P and $\lambda \neq 1$ then λ^{n_0} is ev of P^{n_0}. By PF $|\lambda^n| < 1$, therefore $|\lambda| < 1$. We conclude that P satisfies 1) - 3) $\blacksquare$

Stationary distribution and long-run behavior

Prop. 7.1 Let (X_n) be a MC with finite state space S.

Suppose that there exists $n_0 \in \mathbb{N}$ s.t $[P^{n_0}]_{ij} > 0$ for all $i, j \in S$.

Then for each j, $\pi(j)$ is equal to the asymptotic expected fraction of time the chain spends in j, i.e.,

$$\lim_{n\to\infty} \mathbb{E}\left[\frac{1}{n+1}\sum_{k=0}^{n} \mathbb{1}_{\{X_k=j\}}\right] = \pi(j)$$

Proof.

$$\mathbb{E}\left[\frac{1}{n+1}\sum_{k=0}^{n}\mathbb{1}_{\{X_k=j\}}\right] = \frac{1}{n+1}\sum_{k=0}^{n}\mathbb{P}[X_k=j] = \frac{1}{n+1}\sum_{k=0}^{n}\sum_{i\in S}\mathbb{P}[X_k=j \mid X_0=i]\,\mathbb{P}[X_0=i]$$

$$= \frac{1}{n+1}\sum_{k=0}^{n}[\pi_0 P^k]_j$$

By Cor. 6.6, $[\pi_0 P^k]_j \to \pi(j)$ as $k\to\infty$, for all $j\in S$ and π_0.

Therefore, $\frac{1}{n+1}\sum_{k=0}^{n}[\pi_0 P^k]_j \to \pi(j)$ ■ $\left[\text{if } a_n \to a,\ n\to\infty,\ \text{then } \frac{1}{n}\sum_{k=1}^{n} a_k \xrightarrow[n\to\infty]{} a\right]$