MATH 285: Stochastic Processes

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Today: Reducible Markov chains with finite state space
Markov chains with infinite state space

- Homework 2 is due on Friday, January 21 11:59 PM

General form of transition matrix with finite S

$$P = \left[\begin{array}{cccc|c} P_1 & & & 0 & \\ & P_2 & & & \\ & & P_3 & & 0 \\ 0 & & & \ddots & \\ & & & P_r & \\ \hline & & S & & Q \end{array}\right]$$

P_ℓ submatrix for the recurrent class R_ℓ

P_ℓ is a stochastic matrix, we can consider it as a Markov chain on R_ℓ

$[S \mid Q]$ transition probabilities starting from transient states.

- If P_ℓ is aperiodic, then $P_\ell^n \to \begin{bmatrix} \pi^{(\ell)} \\ \vdots \\ \pi^{(\ell)} \end{bmatrix}$, $n \to \infty$
- What about transient states?
- What if P_ℓ is not aperiodic?

Transient states

Suppose there exists one transient class T

$$\begin{array}{c} \\ \\ \\ \\ \\ T\{ \end{array}\begin{bmatrix} \begin{matrix} P_1 & & & & 0 \\ & P_2 & & & \\ & & P_3 & & \\ 0 & & & \ddots & \\ & & & & P_r \end{matrix} & 0 \\ S & Q \end{bmatrix}$$

- $S \neq 0$

  If $S = 0$ then T is recurrent

- If $S \neq 0$, then Q is substochastic,

  i.e., $\exists\ i \in T$ s.t. $\sum_{j \in T} Q < 1$

- If Q is substochastic, then for all eigenvalues λ of Q $|\lambda| < 1$

  $\Rightarrow Q^n \to 0,\ n \to \infty$, i.e. for $i, j \in T$ $\mathbb{P}_i[X_n = j] \to 0,\ n \to \infty$

- $I + Q + Q^2 + \cdots = I + VDV^{-1} + VD^2V^{-1} + \cdots = V(I + D + D^2 + \cdots)V^{-1}$ converges

  For $i, j \in T$, $\mathbb{E}_i\left[\sum_{k=0}^{\infty} \mathbb{1}_{X_k = j}\right] = \sum_{k=0}^{\infty} \mathbb{P}_i[X_k = j] = p_0(i,j) + p_1(i,j) + p_2(i,j) + \cdots$

  $= \delta_{ij} + Q_{ij} + [Q^2]_{ij} + \cdots = [(I + Q + Q^2 + \cdots)]_{ij} = [(1 - Q)^{-1}]$

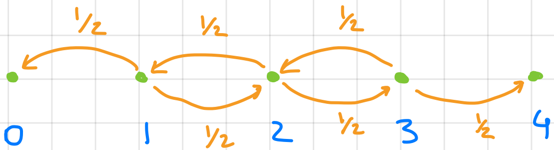
Transient states

Conclusion: if $T \subset S$ is a transient class, then $\forall\, i, j \in T$

$$\lim_{n \to \infty} \mathbb{P}_i [X_n = j] = 0$$

$$\mathbb{E}_i\Big[\sum_{k=0}^{n} \mathbb{1}_{\{X_n = j\}} \Big] = [(I-Q)^{-1}]_{ij}$$ expected number of visits to j starting from i

Example 8.1



$$P = \begin{array}{c} \\ 0 \\ 4 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 4 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{array}\right] \end{array}$$

$$Q = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \qquad (I-Q)^{-1} = \begin{bmatrix} \frac{3}{2} & 1 & \frac{1}{2} \\ 1 & 2 & 1 \\ \frac{1}{2} & 1 & \frac{3}{2} \end{bmatrix}$$

Expected number of visits to (2) starting from (1) is 1

Expected number of steps before absorption starting from (1) is $\frac{3}{2} + 1 + \frac{1}{2} = 3$

Transient states

Recall, First step analysis for the mean hitting time

$$g_i = \mathbb{E}_i[\tau_A] = \begin{cases} 0, & i \in A \\ 1 + \sum_{j \in S} p(i,j) g_j, & i \notin A \end{cases}$$

$$\tau_A = \sum_{n=0}^{\infty} \mathbb{1}_{\{X_n \notin A\}}$$

Instead of adding 1 for each step, add 1 only when X_n visits j:

Denote $S \setminus A =: T$, and for $i, j \in T$ $\quad g_{ij} = \mathbb{E}_i\left[\sum_{n=0}^{\infty} \mathbb{1}_{\{X_n = j\}}\right]$

Then FSA $\quad g_{kj} = 0 \quad$ if $k \in A$

$$g_{ij} = \delta_{ij} + \sum_{k \in S} p(i,k) g_{kj} = \delta_{ij} + \sum_{k \in T} p(i,k) g_{kj}$$

$G = [g_{ij}]$, then $\quad G = I + QG \Rightarrow G = (I - Q)^{-1}$

Transient states

Starting from T, in which class will (X_n) end up ?

Collapse each R_ℓ into one state r_ℓ,
keep transient states t_ℓ, $T = \{t_\ell\}$
$(\tilde{X}_n)$ new MC on the reduced state space, and transition matrix $\tilde{P}$,
with $\tilde{s}(t_i, r_j) = \mathbb{P}_{t_i}[X_1 \in R_j]$

Denote $\tilde{A} = [\alpha(t_i, r_j)]$ with

$\alpha(t_i, r_j) := \mathbb{P}_{t_i}[(X_n) \text{ enters } r_j \text{ eventually}]$

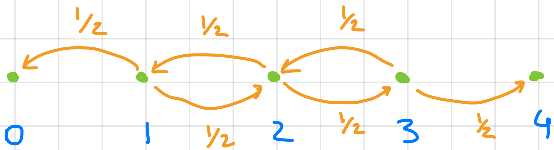
Then $\tilde{A} = (I - Q)^{-1} \tilde{s}$

$$\begin{array}{c} R_1 \\ R_2 \\ \\ \\ \\ T \end{array}
\left[\begin{array}{ccccc|c}
P_1 & & & & & \\
& P_2 & & 0 & & \\
& & P_3 & & & 0 \\
& 0 & & \ddots & & \\
& & & & P_r & \\
\hline
& & S & & & Q
\end{array}\right]$$

$$\tilde{P} = \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ \vdots \\ \\ \\ t_1 \\ t_2 \end{array}
\left[\begin{array}{ccccc|c}
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 & & & & 0 & 0 \\
0 & 0 & 0 & \cdots & 1 & 0 \\
\hline
& & \tilde{S} & & & Q \\
& & & & &
\end{array}\right]$$

Transient states

Example 8.2



$$P = \begin{array}{c} \\ 0 \\ 4 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 4 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{array}\right] \end{array}$$

$\tilde{S}$ $\quad$ Q

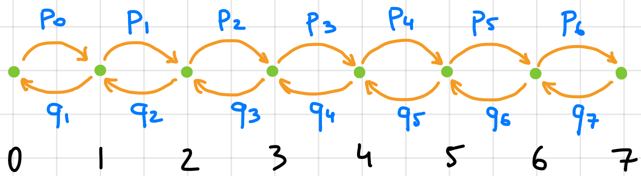
What is the probability that starting from a transient state i we end up in a recurrent state j?

Use $\tilde{A} = (I-Q)^{-1}\tilde{S}$ (nothing to collapse in this case)

$$\tilde{A} = \begin{bmatrix} \frac{3}{2} & 1 & \frac{1}{2} \\ 1 & 2 & 1 \\ \frac{1}{2} & 1 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

- Expected transit times from i to j (think about j as absorbing)...

Birth and death processes (infinite state space)



$S = \{0, 1, 2, 3, \dots\}$

$p(i, i+1) = p_i, \quad p(i, i-1) = 1 - p_i \overset{\text{"}}{=} q_i$

$p(0,1) = p_0, \quad p(0,0) = 1 - p_0$

$p_0 \in [0,1]$, $p_0 = 0$ absorbing, $p_0 = 1$ reflecting

Model of population growth : X_n = size of the population at time n

$\mathbb{P}_i[\exists n \geq 0 : X_n = 0]$ – extinction probability

$\mathbb{P}_i[X_n \to \infty \text{ as } n \to \infty]$ – probability that population explodes

Denote $h(i) := \mathbb{P}_i[\exists n \geq 0 \ X_n = 0] = \mathbb{P}_i[\tau_0 < \infty]$, $\tau_0 = \min\{n \geq 0, X_n = 0\}$

First step analysis:

<u>Theorem 7.0</u> $(h(0), h(1), \dots)$ is the minimal solution to

$$\begin{cases} h(0) = 1 \\ h(i) = \sum_{j=0}^{\infty} p(i,j)\, h(j) \end{cases}$$

Birth and death processes

$$(*)\begin{cases} h(i) = \sum_{j\geq 0} p(i,j)\, h(j) \\ h(0) = 1 \end{cases}$$

$$p(i,j) = \begin{cases} p_i, & j = i+1 \\ q_i, & j = i-1 \\ 0, & \text{otherwise} \end{cases}$$

$$(*)\begin{cases} h(0) = 1 \\ h(1) = p_1 h(2) + q_1 h(0) \\ h(2) = p_2 h(3) + q_2 h(1) \\ \vdots \\ h(i) = p_i h(i+1) + q_i h(i-1) \\ \vdots \end{cases}$$

$$\begin{cases} u(1) = h(1) - h(0) = h(1) - 1 \\ u(2) = \frac{q_1}{p_1} u(1) \\ u(3) = \frac{q_2}{p_2} u(2) = \frac{q_2 q_1}{p_2 p_1} u(1) \\ \vdots \\ u(i+1) = \frac{q_i}{p_i} u(i) = \frac{q_i \cdots q_1}{p_i \cdots p_1} u(1) \\ \vdots \end{cases}$$

$$p_i \underbrace{\left(h(i) - h(i+1)\right)}_{= u(i+1)} = q_i \underbrace{\left(h(i-1) - h(i)\right)}_{= u(i)}$$

$$u(i+1) = \frac{q_i}{p_i} u_i$$

Denote

$$\rho_i := \frac{q_i \cdots q_1}{p_i \cdots p_1}$$

Birth and death processes

$$\begin{cases} u(1) = u(1) \\ u(2) = \rho_1 u(1) \\ u(3) = \rho_2 u(1) \\ \vdots \\ u(i+1) = \rho_i u(1) \\ \vdots \end{cases}$$

$u(i) = h(i-1) - h(i)$

Take the sum of the first i equations

$$h(0) - h(i) = (1 + \rho_1 + \rho_2 + \cdots + \rho_{i-1}) u(1)$$

By Thm. 7.0 we need the minimal solution to (*)

Notice that $u(1)$ uniquely determines all $h(i)$

$$h(i) = h(0) - (1 + \rho_1 + \rho_2 + \cdots + \rho_i) u(1)$$

and the minimal solution corresponds to maximal $u(1)$

- If $1 + \sum_{i=1}^{\infty} \rho_i = \infty$, then $u(1) = 0$ (otherwise $h(0) - h(1) > 1$)

  In this case $h(0) - h(i) = 0 \ \forall i \Rightarrow h(i) = 1$ for all i

  no chance of survival

Birth and death processes

- If $1+\sum_{i=1}^{\infty} \rho_i < \infty$, then for any $a \in [0, \frac{1}{1+\sum_{i=1}^{\infty}\rho_i}]$

we get a solution to (*) by taking

$$\forall i \quad h(0)-h(i) = (1+\rho_1+\rho_2+\cdots+\rho_{i-1})\, a$$

If $u(1) > \frac{1}{1+\sum_{i=1}^{\infty}\rho_i}$, then for some m large enough

$$1 < \left(1+\sum_{i=1}^{m}\rho_i\right) u(1) = h(0)-h(m) \leq 1$$

Therefore, $u(1) = \frac{1}{1+\sum_{i=1}^{\infty}\rho_i}$ is the maximal allowable value of $u(1)$, and the corresponding minimal solution is $h(j) = 1-\left(1+\sum_{i=1}^{j-1}\rho_i\right)\Big/\left(1+\sum_{i=1}^{\infty}\rho_i\right) = \sum_{i=j}^{\infty}\rho_i \Big/ 1+\sum_{i=1}^{\infty}\rho_i$ $\blacksquare$