MATH 285: Stochastic Processes

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Today: Reducible Markov chains with finite state space
Markov chains with infinite state space

- Homework 2 is due on Friday, January 21 11:59 PM

General form of transition matrix with finite S

$$P = \left[\begin{array}{cccc|c} P_1 & & & & \\ & P_2 & & 0 & \\ & & P_3 & & 0 \\ & 0 & & \ddots & \\ & & & P_r & \\ \hline & & S & & Q \end{array}\right]$$

P_ℓ submatrix for the recurrent class R_ℓ

P_ℓ is a stochastic matrix, we can consider it as a Markov chain on R_ℓ

$[S|Q]$ transition probabilities starting from transient states.

- If P_ℓ is aperiodic, then $P_\ell^n \to \begin{bmatrix} \pi^{(\ell)} \\ \vdots \\ \pi^{(\ell)} \end{bmatrix}$, $n \to \infty$
- What about transient states?
- What if P_ℓ is not aperiodic ?

Transient states

Suppose there exists one transient class T

$$\begin{array}{c} \\ \\ \\ \\ T\{ \end{array} \left[\begin{array}{cccc|c} P_1 & & 0 & & \\ & P_2 & & & 0 \\ & & P_3 & & \\ 0 & & & \ddots & \\ & & & P_r & \\ \hline & & S & & Q \end{array}\right]$$

- $S \neq 0$

If $S = 0$ then T is recurrent

- If $S \neq 0$, then Q is substochastic,

i.e., $\exists\; i \in T$ s.t. $\sum_{j \in T} Q < 1$

- If Q is substochastic, then for all eigenvalues λ of Q $|\lambda| < 1$

$\Rightarrow Q^n \to 0,\ n \to \infty$, i.e. for $i, j \in T$ $\mathbb{P}_i[X_n = j] \to 0,\ n \to \infty$

- $I + Q + Q^2 + \dots = I + VDV^{-1} + VD^2V^{-1} + \dots = V(I + D + D^2 + \dots)V^{-1}$ converges

For $i, j \in T$, $\mathbb{E}_i\left[\sum_{k=0}^{n} \mathbb{1}_{X_k = j}\right] =$

$=$

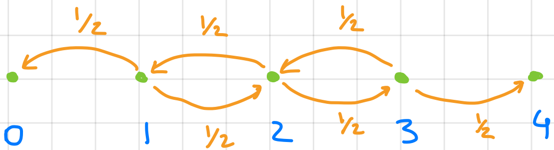
Transient states

Conclusion: if $T \subset S$ is a transient class, then $\forall\, i, j \in T$

$$\lim_{n \to \infty} \mathbb{P}_i [X_n = j] =$$

$$\mathbb{E}_i \Big[\sum_{k=0}^{n} \mathbb{1}_{\{X_n = j\}} \Big] = \quad \text{expected number of visits to } j \text{ starting from } i$$

Example 8.1



$$P = \begin{array}{c} \\ 0 \\ 4 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 4 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix} \end{array}$$

$$Q = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \qquad (I - Q)^{-1} = \begin{bmatrix} \frac{3}{2} & 1 & \frac{1}{2} \\ 1 & 2 & 1 \\ \frac{1}{2} & 1 & \frac{3}{2} \end{bmatrix}$$

Expected number of visits to ② starting from ① is 1

Expected number of steps before absorption starting from ① is $\frac{3}{2} + 1 + \frac{1}{2} = 3$

Transient states

Recall, First step analysis for the mean hitting time

$$g_i = \mathbb{E}_i[\tau_A] = \begin{cases} 0, & i \in A \\ 1 + \sum_{j \in S} P(i,j)\, g_j, & i \notin A \end{cases}$$

$$\tau_A = \sum_{n=0}^{\infty} \mathbb{1}_{\{X_n \notin A\}}$$

Instead of adding 1 for each step, add 1 only when X_n visits j:

Denote $S \setminus A =: T$, and for $i, j \in T$ $g_{ij} =$

Then FSA $g_{ij} =$ if $i \in A$

$g_{ij} =$

$G = [g_{ij}]$, then

Transient states

Starting from T, in which class will (X_n) end up?

$$\begin{array}{c} R_1 \\ R_2 \\ \\ \\ \\ T \end{array} \left[\begin{array}{ccccc|c} P_1 & & & 0 & & \\ & P_2 & & & & 0 \\ & & P_3 & & & \\ & 0 & & \ddots & & \\ & & & & P_r & \\ \hline & & S & & & Q \end{array}\right]$$

Collapse each R_ℓ into one state r_ℓ, keep transient states t_ℓ, $T=\{t_\ell\}$

$(\tilde{X}_n)$ new MC on the reduced state space, and transition matrix $\tilde{P}$,

with $\tilde{s}(t_i, r_j) =$

$$\tilde{P} = \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ \\ \\ t_1 \\ t_2 \end{array} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \hline & & \tilde{S} & & & Q \\ & & & & & \end{array}\right]$$

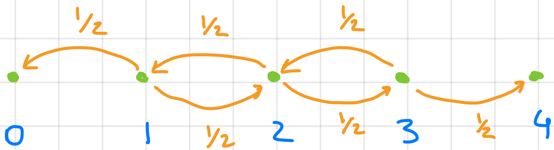
Denote $\tilde{A} = [\alpha(t_i, r_j)]$ with

$\alpha(t_i, r_j) :=$

Then $\tilde{A} =$

Transient states

Example 8.2



$$P = \begin{array}{c} \\ 0 \\ 4 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 4 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{array}\right] \end{array}$$

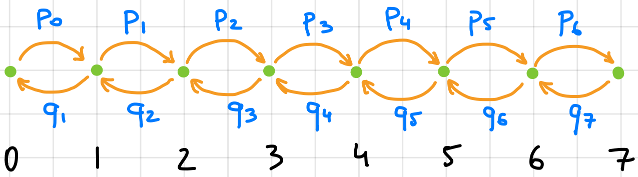
What is the probability that starting from a transient state i we end up in a recurrent state j?

Use $\tilde{A} =$ (nothing to collapse in this case)

$$\tilde{A} = \begin{bmatrix} \frac{3}{2} & 1 & \frac{1}{2} \\ 1 & 2 & 1 \\ \frac{1}{2} & 1 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

- Expected transit times from i to j (think about j as absorbing)...

Birth and death processes (infinite state space)



$S = \{0, 1, 2, 3, \dots\}$

$p(i, i+1) = p_i, \quad p(i, i-1) = 1 - p_i \;\; \text{// } q_i$

$p(0,1) = p_0, \quad p(0,0) = 1 - p_0$

$p_0 \in [0,1]$, $p_0 = 0$ absorbing, $p_0 = 1$ reflecting

Model of population growth : X_n = size of the population at time n

$\mathbb{P}_i[\exists n \geq 0 : X_n = 0]$ – extinction probability

$\mathbb{P}_i[X_n \to \infty \text{ as } n \to \infty]$ – probability that population explodes

Denote $h(i) := \mathbb{P}_i[\exists n \geq 0 \;\; X_n = 0] =$

First step analysis:

<u>Theorem 7.0</u> $(h(0), h(1), \dots)$ is the minimal solution to
$$\begin{cases} h(0) = 1 \\ h(i) = \sum_{j=0}^{\infty} p(i,j)\, h(j) \end{cases}$$

Birth and death processes

$$(*)\begin{cases} h(i) = \sum_{j\geq 0} p(i,j)\, h(j) \\ h(0) = 1 \end{cases}$$

$$p(i,j) = \begin{cases} p_i\,, & j = i+1 \\ q_i\,, & j = i-1 \\ 0\,, & \text{otherwise} \end{cases}$$

$$(*)\begin{cases} h(0) = 1 \\ h(1) = p_1 h(2) + q_1 h(0) \\ h(2) = p_2 h(3) + q_2 h(1) \\ \vdots \\ h(i) = p_i h(i+1) + q_i h(i-1) \\ \vdots \end{cases}$$

$$\begin{cases} u(1) = h(1) - h(0) = h(1) - 1 \\ u(2) = \dfrac{q_1}{p_1} u(1) \\ u(3) = \dfrac{q_2}{p_2} u(2) = \dfrac{q_2 q_1}{p_2 p_1} u(1) \\ \vdots \\ u(i+1) = \dfrac{q_i}{p_i} u(i) = \dfrac{q_i \cdots q_1}{p_i \cdots p_1} u(1) \\ \vdots \end{cases}$$

Denote

Birth and death processes

$$\begin{cases} u(1) = u(1) \\ u(2) = \rho_1 u(1) \\ u(3) = \rho_2 u(1) \\ \quad \vdots \\ u(i+1) = \rho_i u(1) \\ \quad \vdots \end{cases}$$

$u(i) = h(i-1) - h(i)$

Take the sum of the first i equations

By Thm. 7.0 we need the minimal solution to $(*)$

Notice that $u(1)$ uniquely determines all $h(i)$

$$h(i) = h(0) - (1 + \rho_1 + \rho_2 + \cdots + \rho_i)\, u(1)$$

and the minimal solution corresponds to maximal $u(1)$

- If $1 + \sum_{i=1}^{\infty} \rho_i = \infty$, then

  In this case $h(0) - h(i) = 0 \ \forall i \Rightarrow$

Birth and death processes

- If $1+\sum_{i=1}^{\infty} \rho_i < \infty$, then for any

we get a solution to $(*)$ by taking

$$\forall i \quad h(0) - h(i) = (1+\rho_1+\rho_2+\cdots+\rho_{i-1})\, a$$

If $u(1) > \frac{1}{1+\sum_{i=1}^{\infty}\rho_i}$, then for some m large enough

Therefore,

value of $u(1)$, and the corresponding minimal

solution is $h(j) =$

Positive and null recurrence

Let (X_n) be a Markov chain, and let i be a recurrent state. Starting from i, (X_n) revisits i infinitely many times, $\mathbb{P}_i[X_n = i \text{ for infinitely many } n] = 1$

How often does (X_n) revisit state i?

(i) After n steps, (X_n) revisits $i \approx \frac{n}{2}$ times, spends half of the time at i

(ii) After n steps, (X_n) revisits $i \approx \sqrt{n}$ times, the fraction of time spent at i tend to 0 as $n \to \infty$, $\frac{\sqrt{n}}{n} \to 0$, $n \to \infty$

Def 9.2 Let i be a recurrent state for MC (X_n).

Denote $T_i = \min\{n \geq 1 : X_n = i\}$. If , then we call i

If , the we call i

Positive and null recurrence

<u>Remark</u> If i is recurrent, then $\mathbb{P}_i[T_i] < \infty$. But it is still possible that $\mathbb{E}[T_i] = \infty$ or that $\mathbb{E}[T_i] < \infty$.

Example: $Y_1, Y_2 \in \mathbb{N}$, $\mathbb{P}[Y_1 = k] =$, $Y_2 =$, $\mathbb{P}[Y_2 = 2^k] =$.

$$\mathbb{P}[Y_1 < \infty] = \mathbb{P}[Y_2 < \infty] = 1, \quad \mathbb{E}[Y_1] = \quad , \mathbb{E}[Y_2] =$$

<u>Prop 9.4</u> In a finite-state irreducible Markov chain all states are

<u>Proof</u>. Fix state $j \in S$

(1) There exist $N \in \mathbb{N}$ and $q \in (0,1)$ such that for any $i \in S$

(probability of reaching j from i in the next N steps)

Since (X_n) is irreducible,

Take

Positive and null recurrence

(2) For any $i \in S$ $\quad \mathbb{P}_i[T_j > N] \leq$. $\quad$ | follows from (1)

(3) For any $k \in \mathbb{N}$, $\quad \mathbb{P}_j[T_j > (k+1)N] \leq$

For any $i \in S$ $\quad \mathbb{P}_j[T_j > (k+1)N \mid T_j > kN, X_{kN} = i] \overset{(SMP)}{=}$

$\mathbb{P}_j[T_j > (k+1)N] =$

$=$

$=$

$\leq$

Now repeat k times.

Positive and null recurrence

(4) $\mathbb{E}_j[T_j] = \sum_{n=1}^{\infty} \mathbb{P}_j[T_j \geq n] =$

(5) $\mathbb{P}_j[T_j \geq n]$ is

Therefore $\forall\ n \in \{kN+1, \dots, (k+1)N\}$

$$\mathbb{P}_j[T_j \geq n] \leq$$

(6) $\sum_{n=kN+1}^{(k+1)N} \mathbb{P}_j[T_j \geq n] \leq$

Finally, $\mathbb{E}_j[T_j] \leq$

Conclusion: All states of an irreducible MC with finite state space are positive recurrent.

Positive recurrence and stationary distributions

Thm 9.6 Let (X_n) be a Markov chain with a state space that is countable (but not necessarily finite).
Suppose there exists a positive recurrent state $i \in S$, $\mathbb{E}_i[T_i] < \infty$.
For each state $j \in S$ define

$$\gamma(i,j) =$$

(the expected number of visits to j before reaching i).
Then the function $\pi : S \to [0,1]$

$$\pi(j) =$$

is a stationary distribution for (X_n).