MATH 10C: Calculus III (Lecture B00)

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Today: Projectile motion. Functions of two variables

Next: Strang 4.1

Week 5:

- homework 4 (due Friday, October 27)
- regrades of Midterm 1 on Gradescope until October 29

Properties of derivatives of vector-valued functions

Thm 3.3. Let $\vec{r}(t)$ and $\vec{u}(t)$ be differentiable vector-valued functions, let $f(t)$ be a differentiable scalar function, let c be a scalar.

(i) $\frac{d}{dt}[c\vec{r}(t)] = c\vec{r}\,'(t)$ (scalar multiple)

(ii) $\frac{d}{dt}[\vec{r}(t) \pm \vec{u}(t)] = \vec{r}\,'(t) \pm \vec{u}\,'(t)$ (sum and difference)

(iii) $\frac{d}{dt}[f(t)\vec{r}(t)] = f'(t)\vec{r}(t) + f(t)\vec{r}\,'(t)$ (product with scalar function)

(iv) $\frac{d}{dt}[\vec{r}(t) \cdot \vec{u}(t)] = \vec{r}\,'(t) \cdot \vec{u}(t) + \vec{r}(t) \cdot \vec{u}\,'(t)$ (dot product)

(v) $\frac{d}{dt}[\vec{r}(t) \times \vec{u}(t)] = \vec{r}\,'(t) \times \vec{u}(t) + \vec{r}(t) \times \vec{u}\,'(t)$ (cross product)

(vi) $\frac{d}{dt}[\vec{r}(f(t))] = \vec{r}\,'(f(t)) \cdot f'(t)$ (chain rule)

Properties of derivatives of vector-valued functions

(vii) If $\underbrace{\vec{r}(t)\cdot\vec{r}(t)}^{\|\vec{r}(t)\|^2} = c$, then $\vec{r}(t)\cdot\vec{r}\,'(t) = 0$

<u>Proof</u> (iv) $\frac{d}{dt}[\vec{r}(t)\cdot\vec{u}(t)]$

$$= \frac{d}{dt}\left[r_1(t)u_1(t) + r_2(t)u_2(t) + r_3(t)u_3(t)\right]$$

$$= r_1'(t)u_1(t) + r_1(t)u_1'(t)$$

$$+ r_2'(t)u_2(t) + r_2(t)u_2'(t)$$

$$+ r_3'(t)u_3(t) + r_3(t)u_3'(t)$$

$$= \vec{r}\,'(t)\cdot\vec{u}(t) + \vec{r}(t)\cdot\vec{u}\,'(t)$$

(vii) $0 = \frac{d}{dt}[\vec{r}(t)\cdot\vec{r}(t)] = \vec{r}\,'(t)\cdot\vec{r}(t) + \vec{r}(t)\cdot\vec{r}\,'(t) = 2\,\vec{r}(t)\cdot\vec{r}\,'(t)$

This means that if $\|\vec{r}(t)\|$ is constant, then $\vec{r}(t) \perp \vec{r}\,'(t)$

Motion in space

If $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ is the position of the particle at time t, then

- $\vec{v}(t) = \vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle$ is the velocity, and
- $\vec{a}(t) = \vec{r}\,''(t) = \langle x''(t), y''(t), z''(t) \rangle$ is the acceleration, and
- $v(t) = \|\vec{v}(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$ is the speed

Example: Projectile motion

Consider an object moving with initial velocity $\vec{v}_0$ but with no forces acting on it other than gravity (ignore the effect of air resistance).

Newton's second law: $\vec{F} = m\vec{a}$, where m = mass of the object

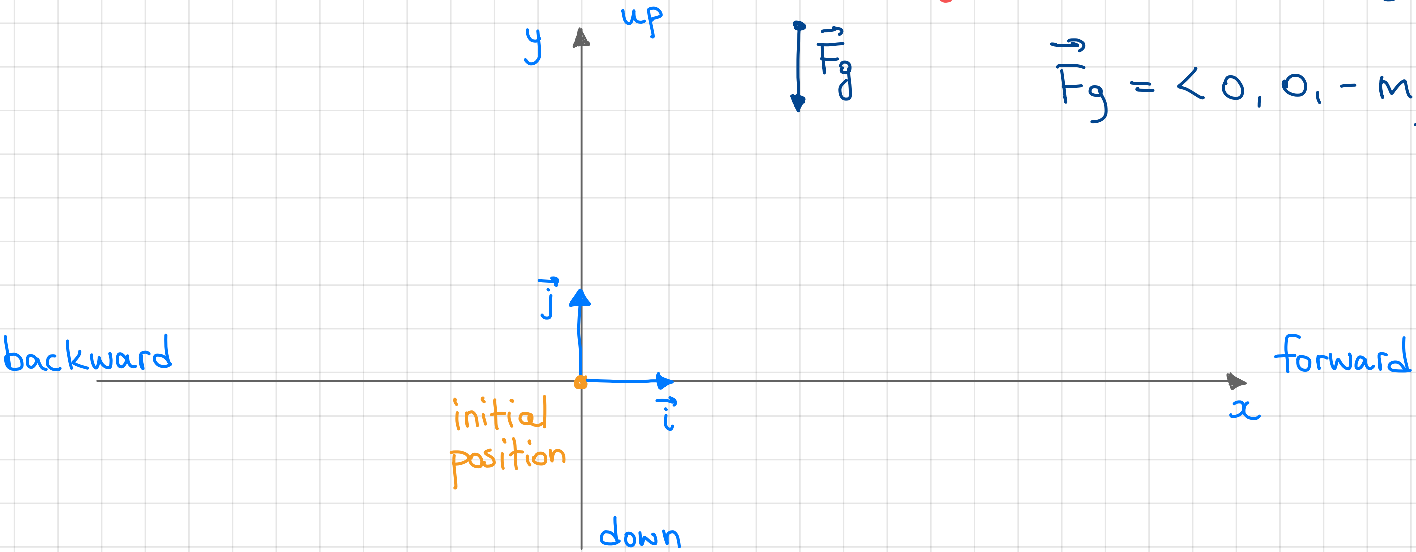
Earth's gravity: $\|\vec{F}_g\| = m \cdot g$, where $g \approx 9.8\ m/s^2$

Projectile motion

Fix the coordinate system:

$$\vec{F}_g = -mg\vec{j} = \langle 0, -mg \rangle$$

$$\vec{F}_g = \langle 0, 0, -mg \rangle$$



Newton's second law: $\vec{F} = m\vec{a}$

Earth's gravity: $\vec{F}_g = -mg\vec{j}$

Earth's gravity is the only force acting on the object

$$\vec{F} = \vec{F}_g$$

Projectile motion

$$\vec{F}(t) = \vec{F}_g : \qquad m\vec{a}(t) = -mg\cdot\vec{j}$$

$$\vec{a}(t) = -g\cdot\vec{j} \qquad (\text{constant acceleration})$$

Since $\vec{a}(t) = \vec{v}'(t)$, we have $\vec{v}'(t) = -g\cdot\vec{j}$

Take antiderivative: $\vec{v}(t) = \int -g\cdot\vec{j}\,dt = -gt\cdot\vec{j} + \vec{c}_1$

Determine $\vec{c}_1$ by taking $\vec{v}(0) = \vec{v}_0$ (initial velocity):

$$\vec{v}(0) = -g\cdot 0 + \vec{c}_1 = \vec{c}_1 = \vec{v}_0$$

This gives the velocity of the object:

$$\vec{v}(t) = -gt\cdot\vec{j} + \vec{v}_0$$

Similarly, $\vec{v}(t) = \vec{r}'(t)$. By taking the antiderivative and $\vec{r}(0) = \vec{r}_0$

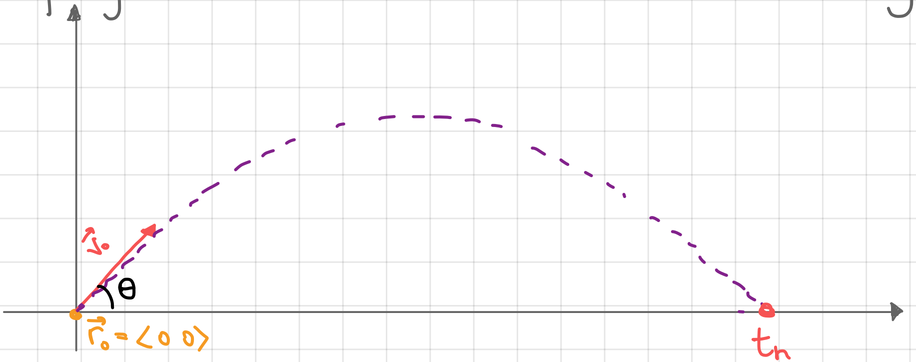
$$\vec{r}(t) = \int \vec{v}(t)\,dt = -g\frac{t^2}{2}\cdot\vec{j} + t\cdot\vec{v}_0 + \vec{c}_0$$

$\vec{r}(0) = \vec{c}_0 = \vec{r}_0$, so $\vec{r}(t) = -g\frac{t^2}{2} + t\vec{v}_0 + \vec{r}_0$

Projectile motion

$\sin(2\theta) = 2\sin\theta\cos\theta$

A projectile is shot by a howitzer with initial speed 800 m/s on a flat terrain. Determine the max distance the projectile can cover before hitting the ground.



Since the initial speed is given, the initial velocity can be determined by the angle: $\vec{v}_0 = 800\langle\cos\theta, \sin\theta\rangle$

Equation of the trajectory: $\vec{r}(t) = -10 \cdot \frac{t^2}{2} \cdot \vec{j} + 800t\cos\theta\,\vec{i} + 800t\sin\theta\,\vec{j}$

Hitting the ground: second component or $\vec{r}(t)$ is 0: $(-5t^2 + 800t\sin\theta) = 0$

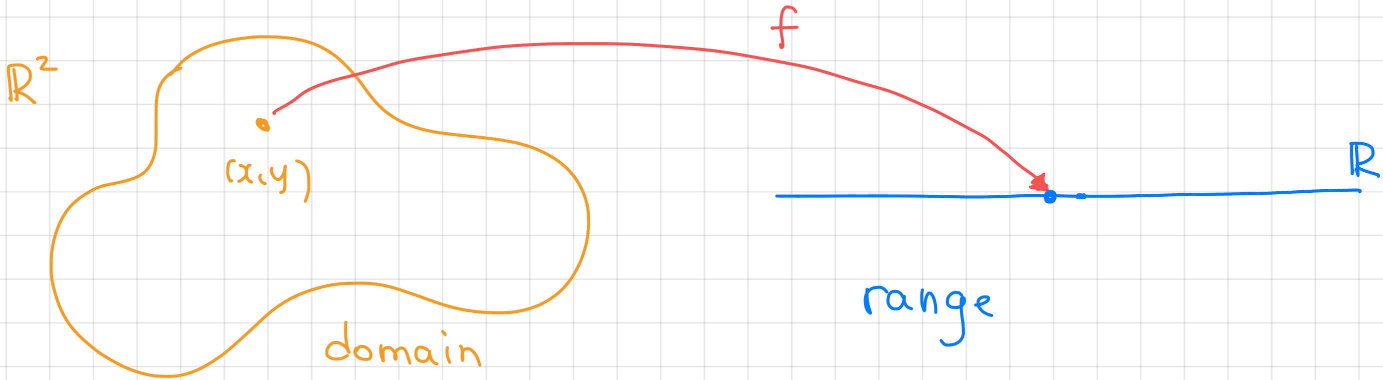
$t(-5t + 800\sin\theta) = 0$, so $t_h = \frac{800\sin\theta}{5} = 160\cdot\sin\theta$. The position of the hit is

$\vec{r}(t_h) = 0\cdot\vec{j} + 800\cdot 160\cdot\sin\theta\cdot\cos\theta\,\vec{i} = 64000\cdot\sin(2\theta)$. Maximum is achieved when $\sin(2\theta) = 1$, i.e., $2\theta = \frac{\pi}{2}$, $\theta = \frac{\pi}{4} = 45°$. Max distance is 64 km.

Functions of several variables

Functions of two variables

<u>Def.</u> A function of two variables maps each ordered pair (x,y) in a subset $D \subset \mathbb{R}^2$ to a unique real number $z = f(x,y)$. The set D is called the domain of the function. The range of f is the set of all real numbers z that has at least one ordered pair $(x,y) \in D$ s.t. $f(x,y) = z$.



If not specified, we choose the domain to be the set of all pairs (x,y) for which $f(x,y)$ is well-defined.