MATH 10C: Calculus III (Lecture B00)

mathweb.ucsd.edu/~ynemish/teaching/10c

Today: Partial derivatives

Next: Strang 4.4

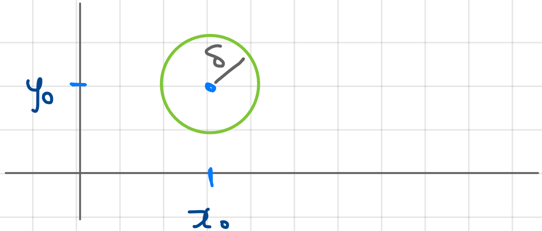
Week 5:

- homework 4 (due Friday, October 28)
- regrades of Midterm 1 on Gradescope until October 30

Limit of a function of two variables

<u>Def</u> Consider a point $(a,b) \in \mathbb{R}^2$. A δ-disk centered at point (a,b) is the open disk of radius δ centered at (a,b)

$$\{(x,y) \mid (x-a)^2+(y-b)^2<\delta^2\}$$



<u>Def</u>. The limit of $f(x,y)$ as (x,y) approaches (x_0,y_0) is L

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

if for each $\varepsilon>0$ there exists a small enough $\delta>0$ such that all points in a δ-disk around (x_0,y_0), except possible (x_0,y_0) itself, $f(x,y)$ is no more than ε away from L. (For any $\varepsilon>0$ there exists $\delta>0$ such that $|f(x,y)-L|<\varepsilon$ whenever $\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta$.)

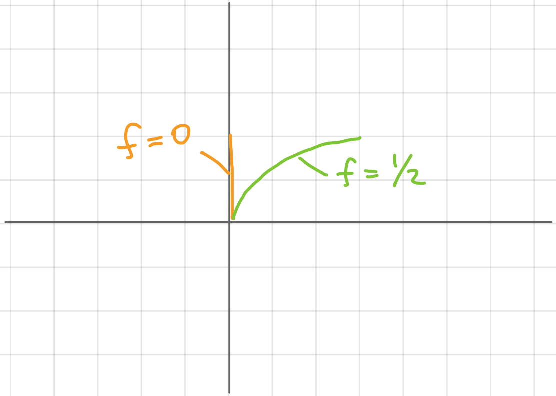
Limit of a function of two variables

This definition ensures that if $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$, then

any way of approaching (x_0,y_0) results in the same limit L.

(Another) example when the limit fails to exist:

$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ does not exist



- approach $(0,0)$ along the line $x=0$; on this line $\frac{0\cdot y^2}{0^2+y^4}=0$

- approach $(0,0)$ along the curve $x=y^2$, $\frac{y^2\cdot y^2}{y^4+y^4}=\frac{1}{2}$

Computing limits. Limit laws

Theorem 4.1 Let $\lim_{(x,y)\to(a,b)} f(x,y) = L$, $\lim_{(x,y)\to(a,b)} g(x,y) = M$, c - constant

- $\lim_{(x,y)\to(a,b)} c = c$
- $\lim_{(x,y)\to(a,b)} x = a$
- $\lim_{(x,y)\to(a,b)} y = b$
- $\lim_{(x,y)\to(a,b)} [f(x,y) \pm g(x,y)] = L \pm M$
- $\lim_{(x,y)\to(a,b)} [f(x,y)g(x,y)] = LM$
- If $M \neq 0$, $\lim_{(x,y)\to(a,b)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}$
- $\lim_{(x,y)\to(a,b)} [c\, f(x,y)] = cL$
- $\lim_{(x,y)\to(a,b)} [f(x,y)]^n = L^n$
- $\lim_{(x,y)\to(a,b)} \sqrt[n]{f(x,y)} = \sqrt[n]{L}$

Computing limits. Limit laws

Examples

$$\lim_{(x,y)\to(0,0)} \frac{xy}{y} = \lim_{(x,y)\to(0,0)} x = 0$$

$$\lim_{(x,y)\to(1,2)} \frac{\sqrt{3x-y}}{(x^2+xy+y^3)^2}$$

$$= \frac{\lim_{(x,y)\to(1,2)} \sqrt{3x-y}}{\lim_{(x,y)\to(1,2)} (x^2+xy+y^3)^2}$$

$$= \frac{\sqrt{\lim_{(x,y)\to(1,2)} [3x-y]}}{\left(\lim_{(x,y)\to(1,2)} [x^2+xy+y^3]\right)^2}$$

$$= \frac{\sqrt{3\lim_{(x,y)\to(1,2)} x - \lim_{(x,y)\to(1,2)} y}}{\left(\left(\lim_{(x,y)\to(1,2)} x\right)^2 + \left(\lim_{(x,y)\to(1,2)} x\right)\left(\lim_{(x,y)\to(1,2)} y\right) + \left(\lim_{(x,y)\to(1,2)} y\right)^3\right)^2}$$

$$= \frac{\sqrt{3\cdot 1 - 2}}{(1^2+1\cdot 2+2^3)^2} = \frac{1}{11^2} = \frac{1}{121}$$

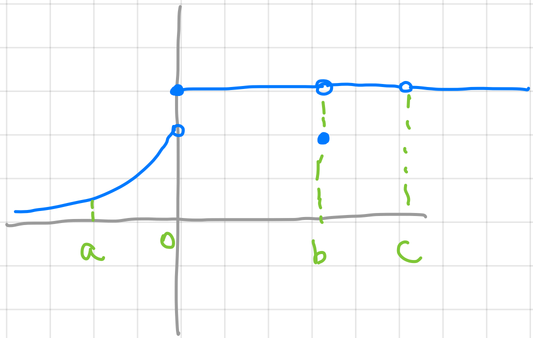
Continuity of functions of two variables

Def. A function $f(x,y)$ is continuous at a point (a,b) if

(i) $f(a,b)$ exists ;

(ii) $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists ; and

(iii) $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$



Properties

1. If $f(x,y)$ and $g(x,y)$ are continuous at (x_0, y_0), then

   $f(x,y) \pm g(x,y)$ is continuous at (x_0, y_0)

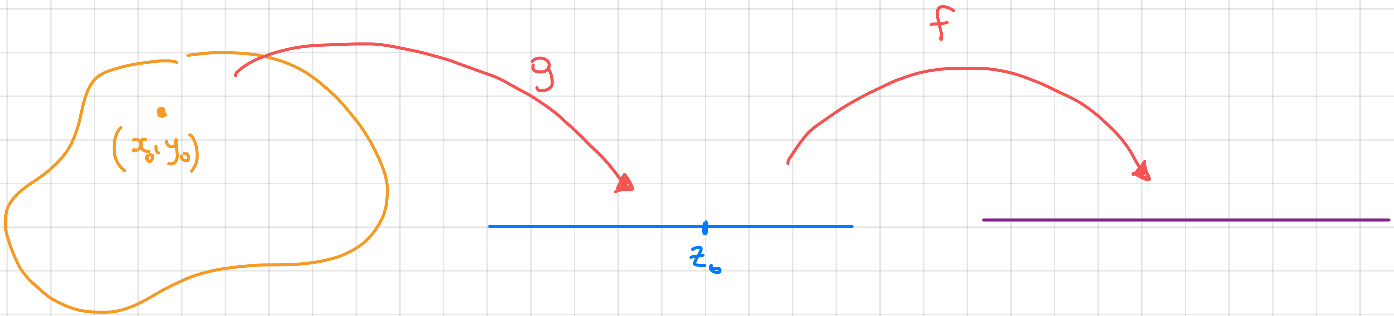
2. If $\varphi(x)$ is continuous at x_0 and $\psi(y)$ is continuous at y_0,

   then $f(x,y) = \varphi(x)\psi(y)$ is continuous at (x_0, y_0)

Continuity of functions of two variables

Properties (cont.)

3. If $g(x,y)$ is continuous at (x_0,y_0), and $f(z)$ is continuous at $z_0 := g(x_0,y_0)$, then $f \circ g\,(x,y) = f(g(x,y))$ is continuous at (x_0,y_0)



Continuity of functions of two variables

Example $\dfrac{\sqrt{3x-y}}{(x^2+xy+y^3)^2}$: $\overset{g(x,y)}{3x-y}$ is continuous on $\mathbb{R}^2$

$f(g(x,y))$ $= f_1(x,y)$ $f(z) = \sqrt{z}$ is continuous for all $z \geq 0$

so $\sqrt{3x-y}$ is continuous for all (x,y) such that $3x-y \geq 0$

Similarly, $g(x,y) = x^2+xy+y^3$ is continuous on $\mathbb{R}^2$

$f(z) = \dfrac{1}{z^2}$ is continuous for all $z \neq 0$

$f(g(x,y))$

so $\dfrac{1}{(x^2+xy+y^3)^2}$ is continuous at all (x,y) such that $x^2+xy+y^3 \neq 0$

$= f_2(x,y)$

Take $(x_0,y_0)=(1,2)$. Then $3\cdot 1-2=1>0$, $1^2+1\cdot 2+2^3=11\neq 0$, so both f_1 and f_2 are continuous at $(1,2)$ and thus

$$\lim_{(x,y)\to(1,2)} f_1(x,y)\, f_2(x,y) = \lim_{(x,y)\to(1,2)} f_1(x,y) \lim_{(x,y)\to(1,2)} f_2(x,y) = f_1(1,2)\, f_2(1,2) = 1 \cdot \frac{1}{121}$$