MATH 10C: Calculus III (Lecture B00)

mathweb.ucsd.edu/~ynemish/teaching/10c

Today: Partial derivatives

Next: Strang 4.4

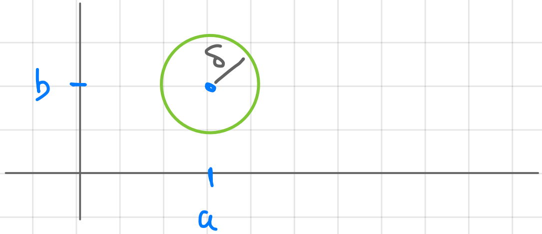
Week 6:

- homework 5 (due Friday, November 4, 11:59 PM)

Limit of a function of two variables (ε, δ error tolerance)

Def Consider a point $(a,b) \in \mathbb{R}^2$. A δ-disk centered at point (a,b) is the open disk of radius δ centered at (a,b)

$$\{ (x,y) \mid (x-a)^2 + (y-b)^2 < \delta^2 \}$$



Def. The limit of $f(x,y)$ as (x,y) approaches (x_0, y_0) is L

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

if for each $\varepsilon > 0$ there exists a small enough $\delta > 0$ such that all points in a δ-disk around (x_0,y_0), except possible (x_0,y_0) itself, $f(x,y)$ is no more than ε away from L. (For any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x,y) - L| < \varepsilon$ whenever $\sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$.)

Computing limits. Limit laws

Theorem 4.1 Let $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$, $\lim\limits_{(x,y)\to(a,b)} g(x,y) = M$, c - constant

- $\lim\limits_{(x,y)\to(a,b)} c = c$
- $\lim\limits_{(x,y)\to(a,b)} x = a$
- $\lim\limits_{(x,y)\to(a,b)} y = b$
- $\lim\limits_{(x,y)\to(a,b)} [f(x,y) \pm g(x,y)] = L \pm M$
- $\lim\limits_{(x,y)\to(a,b)} [f(x,y)g(x,y)] = LM$
- If $M \neq 0$, $\lim\limits_{(x,y)\to(a,b)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}$
- $\lim\limits_{(x,y)\to(a,b)} [c\, f(x,y)] = cL$
- $\lim\limits_{(x,y)\to(a,b)} [f(x,y)]^n = L^n$
- $\lim\limits_{(x,y)\to(a,b)} \sqrt[n]{f(x,y)} = \sqrt[n]{L}$

Examples

$$\lim_{(x,y)\to(0,0)} \frac{xy+1}{x^2+y^2+1} =$$

$$\lim_{(x,y)\to(2,1)} \frac{x-y-1}{\sqrt{x-y}-1} =$$

Partial derivatives of functions of two variables

Functions of one variable $y = f(x)$: the derivative gives the instantaneous rate of change of y as a funtion of x.

Functions of two variables $z = f(x,y)$ have 2 independent variables, we need two (partial) derivatives.

<u>Def</u> The partial derivative of $f(x,y)$ with respect to x is

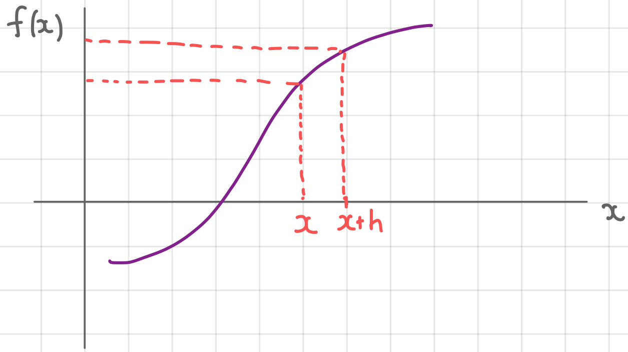
$$f_x = \frac{\partial f}{\partial x} =$$

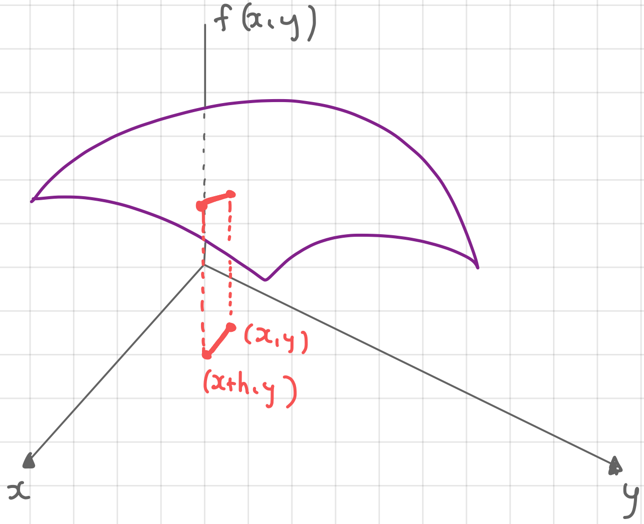
The partial derivative of $f(x,y)$ with respect to y is

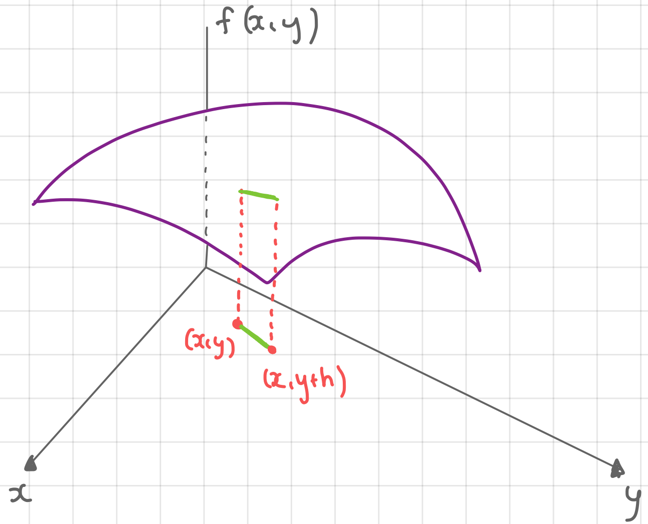
$$f_y = \frac{\partial f}{\partial y} =$$

Partial derivatives of functions of two variables

Partial derivatives measure the instantaneous rate of change of f if we change only the x variable $\frac{\partial f}{\partial x}$ or only the y variable $\frac{\partial f}{\partial y}$







Calculating partial derivatives

Rule To differentiate $f(x,y)$ with respect to x, treat the variable y as a constant, and differentiate f as a function of one variable x:

$$\frac{\partial}{\partial x}\left(x^3 - 12xy^2 - x^2y + 4x - y - 3\right) =$$

To differentiate $f(x,y)$ with respect to y, treat the variable x as a constant, and differentiate f as a function of one variable y:

$$\frac{\partial}{\partial y}\left(x^3 - 12xy^2 - x^2y + 4x - y - 3\right) =$$

Calculating partial derivatives

Example $\quad f(x,y) = e^{-\frac{x^2+y^2}{2}}$

Compute $\quad \dfrac{\partial f}{\partial x} =$

$$\frac{\partial f}{\partial y} =$$

Higher-order partial derivatives

Each partial derivative is itself a function of two variables, so we can compute their partial derivatives, which we call higher-order partial derivatives. For example, there are 4 second-order partial derivatives

$$\frac{\partial^2 f}{\partial x^2} = \qquad , \quad \frac{\partial^2 f}{\partial y \partial x} = \qquad , \quad \frac{\partial^2 f}{\partial x \partial y} = \qquad , \quad \frac{\partial^2 f}{\partial y^2} =$$

f_{xy} and f_{yx} are called

f_{xy} and f_{yx} are not necessarily equal.

Thm If f_{xy} and f_{yx} are continuous on an open disk D, then $f_{xy} = f_{yx}$ on D.

Higher-order partial derivatives

Example Let $f(x,y) = x^2\cos(2x-y) + xe^{-y^2}$

$$\frac{\partial f}{\partial x} =$$

$$\frac{\partial^2 f}{\partial y \partial x} =$$

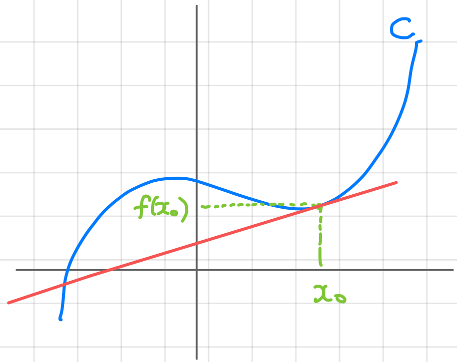
$$\frac{\partial f}{\partial y} =$$

$$\frac{\partial^2 f}{\partial x \partial y} =$$

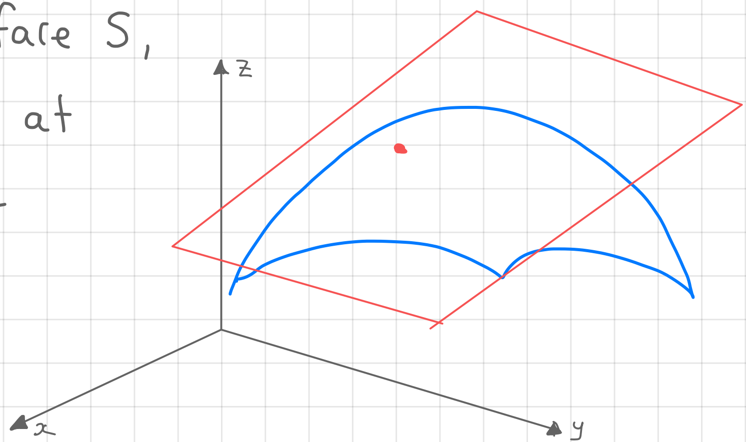
It is not true in general that $f_{xy} = f_{yx}$.

Tangent planes

Recall, if f is a function of one real variable, then its graph determines a curve C in $\mathbb{R}^2$, and the tangent line to the graph of f at point x_0 is the line that "touches" the curve C at point $(x_0, f(x_0))$

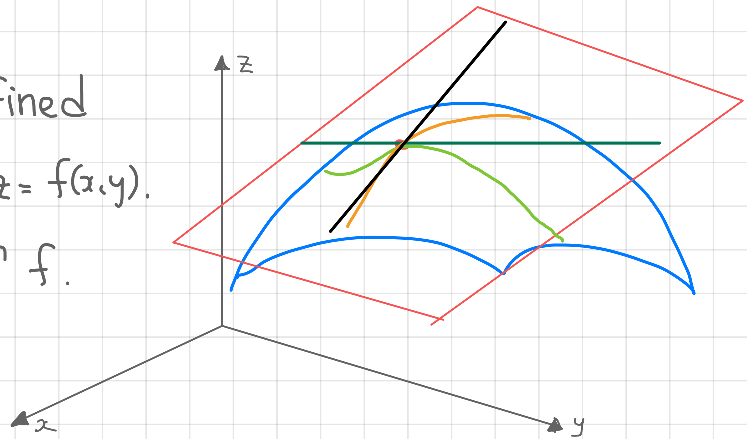


If f is a function of two variables, then its graph determines a surface S, and the tangent plane to S at $(x_0, y_0, f(x_0, y_0))$ is a plane that "touches" S at this point.



Tangent plane

Def. Let $P_0 = (x_0, y_0, z_0)$ be a point on a surface S, and let C be any curve passing through P_0 and lying entirely in S. If the tangent lines to all such curves C at P_0 lie in the same plane, then this plane is called the



Def. Let S be a surface defined by a differentiable function $z = f(x,y)$. Let $P_0 = (x_0, y_0)$ be in the domain of f. Then the equation of the tangent plane to S at P_0 is

Tangent plane

To see that this formula is correct, we can find two curves in S that pass through $(x_0, y_0, f(x_0, y_0))$ and determine the equations of the tangent lines.

Take $\vec{p}(t) =$ and $\vec{q}(s) =$

Then for any t (such that (t, y_0) is in the domain of f) $\vec{p}(t)$. Similarly, for any s $\vec{q}(s)$. Moreover,

Tangent line to $\vec{p}(t)$ at $t = x_0$: $\vec{\ell}_p(t) =$

with $\vec{p}\,'(t) =$

Similarly, tangent line to $\vec{q}(s)$ at $s = y_0$: $\vec{\ell}_q(s) =$

$\vec{q}\,'(s) =$

Tangent plane

Vectors $\vec{p}\,'(x_0) = \langle 1, 0, \frac{\partial f}{\partial x}(x_0, y_0) \rangle$ and $\vec{q}\,'(y_0) = \langle 0, 1, \frac{\partial f}{\partial y}(x_0, y_0) \rangle$ are not parallel, therefore, together with the point $(x_0, y_0, f(x_0, y_0))$ they determine a plane with normal vector

$\vec{n} =$

The equation of a plane passing through $(x_0, y_0, f(x_0, y_0))$ with normal vector $\vec{n}$ is

Tangent plane

Example Find the equation of the tangent plane to the surface defined by the function $f(x,y) = e^{xy}$ at point $(1,-1)$

- Step 1: Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

$$\frac{\partial f}{\partial x} = \qquad\qquad \frac{\partial f}{\partial y} =$$

- Step 2: Evaluate $\frac{\partial f}{\partial x}(x_0, y_0)$ and $\frac{\partial f}{\partial y}(x_0, y_0)$

$$\frac{\partial f}{\partial x}(1,-1) = \qquad\qquad \frac{\partial f}{\partial y}(1,-1) =$$

- Step 3: Evaluate $f(x_0, y_0)$: $f(1,-1) =$

- Step 4: Plug everything into the equation: