MATH 10C: Calculus III (Lecture B00)

mathweb.ucsd.edu/~ynemish/teaching/10c

Today: Tangent planes

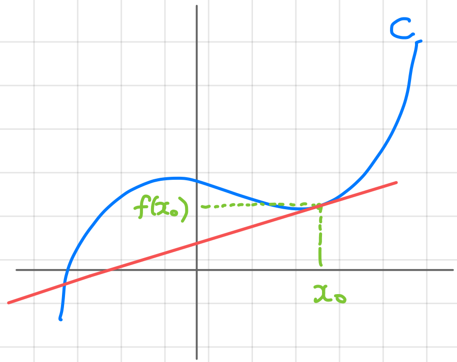
Next: Strang 4.4

Week 6:

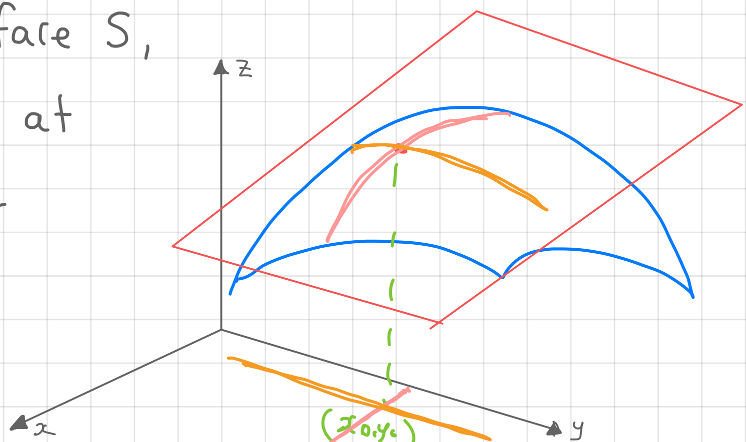
- homework 5 (due Friday, November 4, 11:59 PM)

Tangent planes

Recall, if f is a function of one real variable, then its graph determines a curve C in $\mathbb{R}^2$, and the tangent line to the graph of f at point x_0 is the line that "touches" the curve C at point $(x_0, f(x_0))$



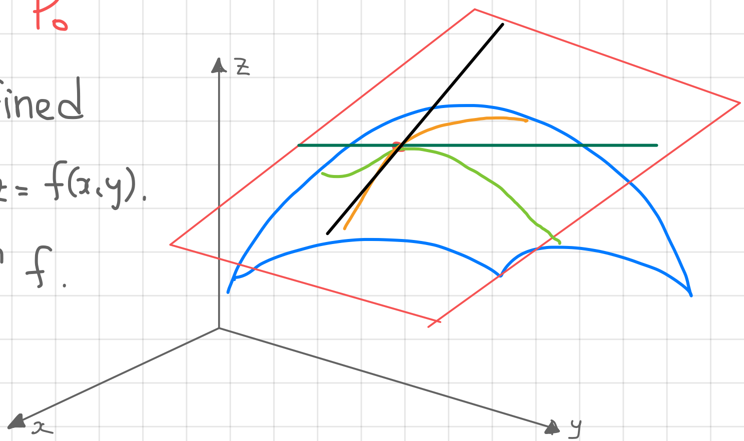
If f is a function of two variables, then its graph determines a surface S, and the tangent plane to S at $(x_0, y_0, f(x_0, y_0))$ is a plane that "touches" S at this point.



Tangent plane

<u>Def</u>. Let $P_0 = (x_0, y_0, z_0)$ be a point on a surface S, and let C be any curve passing through P_0 and lying entirely in S. If the tangent lines to all such curves C at P_0 lie in the same plane, then this plane is called the tangent plane to S a P_0

<u>Def</u>. Let S be a surface defined by a differentiable function $z = f(x,y)$. Let $P_0 = (x_0, y_0)$ be in the domain of f.



Then the equation of the tangent plane to S at P_0 is

$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y)(y - y_0)$$

Tangent plane

To see that this formula is correct, we can find two curves in S that pass through $(x_0, y_0, f(x_0,y_0))$ and determine the equations of the tangent lines.

Take $\vec{p}(t) = \langle t, y_0, f(t,y_0)\rangle$ and $\vec{q}(s) = \langle x_0, s, f(x_0,s)\rangle$

Then for any t (such that (t,y_0) is in the domain of f) $\vec{p}(t)$ is a point on S. Similarly, for any s $\vec{q}(s)$ is a point on S. Moreover, $\vec{p}(x_0) = \vec{q}(y_0) = \langle x_0, y_0, f(x_0,y_0)\rangle$

Tangent line to $\vec{p}(t)$ at $t=x_0$: $\vec{\ell}_p(t) = \vec{p}(x_0) + \vec{p}\,'(x_0)(t-x_0)$

with $\vec{p}\,'(t) = \langle 1, 0, \frac{\partial f}{\partial x}(t,y_0)\rangle$

Similarly, tangent line to $\vec{q}(s)$ at $s=y_0$: $\vec{\ell}_q(s) = \vec{q}(y_0) + \vec{q}\,'(y_0)(s-y_0)$

$$\vec{q}\,'(s) = \langle 0, 1, \frac{\partial f}{\partial y}(x_0,s)\rangle$$

Tangent plane

$\vec{n} = \langle a, b, c \rangle$, $P_0 = (x_0, y_0, z_0)$, $\vec{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

Vectors $\vec{p}\,'(x_0) = \langle 1, 0, \frac{\partial f}{\partial x}(x_0, y_0) \rangle$ and $\vec{q}\,'(y_0) = \langle 0, 1, \frac{\partial f}{\partial y}(x_0, y_0) \rangle$ are not parallel, therefore, together with the point $(x_0, y_0, f(x_0, y_0))$ they determine a plane with normal vector

$$\vec{n} = \vec{q}\,'(y_0) \times \vec{p}\,'(x_0) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & \frac{\partial f}{\partial y}(x_0, y_0) \\ 1 & 0 & \frac{\partial f}{\partial x}(x_0, y_0) \end{vmatrix}$$

$$= \left\langle \frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0), -1 \right\rangle$$

The equation of a plane passing through $(x_0, y_0, f(x_0, y_0))$ with normal vector $\vec{n}$ is

$$\frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) - (z - f(x_0, y_0)) = 0$$

Tangent plane

Example Find the equation of the tangent plane to the surface defined by the function $f(x,y) = e^{xy}$ at point $(1,-1)$

- Step 1: Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

$$\frac{\partial f}{\partial x} = e^{xy} y \qquad \frac{\partial f}{\partial y} = e^{xy} \cdot x$$

- Step 2: Evaluate $\frac{\partial f}{\partial x}(x_0, y_0)$ and $\frac{\partial f}{\partial y}(x_0, y_0)$

$$\frac{\partial f}{\partial x}(1,-1) = e^{-1} \cdot (-1) = -e^{-1} = -\frac{1}{e} \qquad \frac{\partial f}{\partial y}(1,-1) = e^{-1} \cdot 1 = e^{-1} = \frac{1}{e}$$

- Step 3: Evaluate $f(x_0, y_0)$: $f(1,-1) = e^{-1} = \frac{1}{e}$

- Step 4: Plug everything into the equation:

$$z = \frac{1}{e} + \left(-\frac{1}{e}\right)(x-1) + \frac{1}{e}(y+1)$$

Tangent plane does not always exist at every point

Example (tangent plane does not exist at $(0,0)$)

Let $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq 0 \\ 0, & (x,y) = 0 \end{cases}$ ($f(x,y)$ is continuous)

S - surface defined by $f(x,y)$

Consider the curves:

Consider the curve $\vec{p}(t) =$

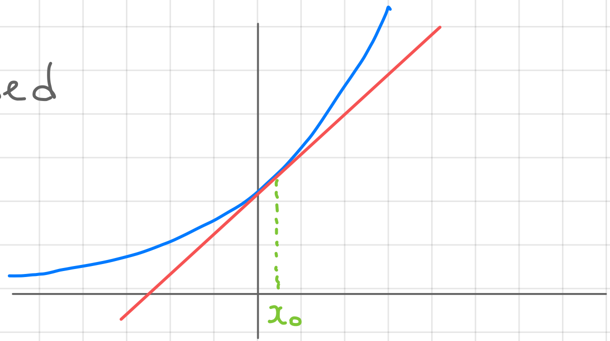
Then $f(t,t) =$

For a tangent plane to a surface to exist, it is sufficient that the function that defines the surface is differentiable.

Linear approximation

Functions of one variable:

$$y = f(x_0) + f'(x_0)(x - x_0)$$



the tangent line at x_0 can be used as the linear approximation of a function $f(x)$ at points x close to x_0:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) \quad \text{for } x \text{ close to } x_0$$

Functions of two variables: the tangent plane at (x_0, y_0) can be used as the linear approximation of $f(x,y)$ at points close to (x_0, y_0)

Def. Given a function $z = f(x,y)$ with continuous partial derivatives that exist at (x_0, y_0), the linear approximation of f at point (x_0, y_0) is given by

$$L(x,y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Linear approximation

Example

Given function $f(x,y) = e^{2x-y-1}$ approximate $f(1.01, 0.99)$ using points $(1,1)$ as (x_0, y_0).

- Compute the derivatives

$$f_x(x,y) = 2e^{2x-y-1} \quad , \quad f_y(x,y) = -e^{2x-y-1}$$

- Evaluate f, f_x and f_y at (x_0, y_0)

$$f(1,1) = e^0 = 1 \; , \quad f_x(1,1) = 2 \; , \quad f_y(1,1) = -1$$

- Write the linear approximation

$$e^{0.03} \approx 1.0304$$

$$L(x,y) = 1 + 2(x-1) - (y-1)$$

- Compute the approximation: $L(1.01, 0.99) = 1 + 2 \cdot 0.01 - (-0.01) = 1.03$