MATH 10C: Calculus III (Lecture B00)

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Today: Differentiability. Chain rule

Next: Strang 4.6

Week 6:

- homework 5 (due Friday, November 4, 11:59 PM)

Differentiability

Functions of one variable: if a function is differentiable at x_0, the graph at x_0 is smooth (no corners), tangent line is well defined and approximates well the function around x_0.

Functions of two variables: differentiability gives the condition when the surface at (x_0, y_0) is smooth, by which we mean that the tangent plane at (x_0, y_0) exists.

Notice, that whenever $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist, we can always write the equation

$$z = f(x_0, y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0). \qquad (*)$$

But this does not mean that the tangent plane exists (if it exists, it is given by $(*)$).

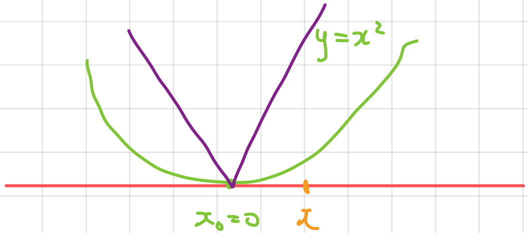
Differentiability

Def. f is differentiable at (x_0, y_0) if $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ exist and the error term

$$E(x,y) = f(x,y) - \left[f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \right]$$

satisfies

$$\lim_{(x,y)\to(x_0,y_0)} \frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$$



This means that

$$f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + E(x,y)$$

and $E(x,y)$ goes to zero faster than the distance between (x,y) and (x_0,y_0).

Remark If $f(x,y)$ is differentiable at (x_0,y_0), then $f(x,y)$ is continuous at (x_0,y_0).

Differentiability

$(x^\alpha)' = \alpha x^{\alpha-1}$ $\quad (-x)^2 = x^2$

The existence of partial derivatives is not sufficient to have differentiability.

Example

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

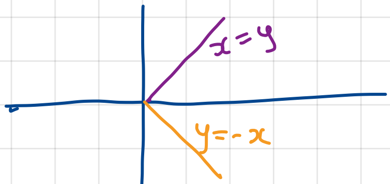
$f(x,y)$ is not differentiable at $(0,0)$

Then $f_x(x,y) = \begin{cases} \dfrac{y^3}{(x^2+y^2)^{3/2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$, $f_y(x,y) = \begin{cases} \dfrac{x^3}{(x^2+y^2)^{3/2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

For $(x_0,y_0) = (0,0)$, $f(0,0) = 0$, $f_x(0,0) = 0$, $f_y(0,0) = 0$, so

$E(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}} - 0$, and $\displaystyle\lim_{(x,y)\to(0,0)} \frac{E(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist

$$f_x(0,0) = \lim_{h\to 0} \frac{\overset{=0}{f(h,0)} - \overset{=0}{f(0,0)}}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$



Differentiability

But, if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist AND are continuous in a neigborhood of (x_0, y_0), then f is differentiable at (x_0, y_0)

Theorem

If $f(x,y)$, $f_x(x,y)$, $f_y(x,y)$ all exist in a neighborhood of (x_0, y_0) and are continuous at (x_0, y_0) then $f(x,y)$ is differentiable at (x_0, y_0).

The chain rule

$f\circ g(x)$

Recall that for functions of one variable

$$\frac{d}{dx}(f(g(x))) = f'(g(x))\,g'(x)$$

Thm (Chain rule for one independent variable)

$f(x(t), y(t))$

Let $x(t)$ and $y(t)$ be differentiable functions, let $f: \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function. Then

$$\frac{d}{dt}[f(x(t), y(t))] = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$

Example Compute $\frac{d}{dt}[f(\underbrace{\sin t}_{x(t)}, \underbrace{\cos t}_{y(t)})]$ with $f(x,y) = 4x^2 + 3y^2$

$$\frac{\partial f}{\partial x} = 8x, \quad \frac{\partial f}{\partial y} = 6y, \quad \frac{d}{dt}\sin t = \cos t, \quad \frac{d}{dt}\cos t = -\sin t$$

$$\frac{d}{dt}[f(\sin t, \cos t)] = 8x(t)\cdot\cos t + 6y(t)(-\sin t) = 8\sin t\cdot\cos t - 6\cos t\cdot\sin t$$

$$= 2\sin t\cos t$$

The chain rule

<u>Thm</u> (Chain rule for two independent variables)

Suppose $x(u,v)$ and $y(u,v)$ are differentiable, and suppose $f(x,y)$ is differentiable. Then $z = f(x(u,v), y(u,v))$ is differentiable (function from $\mathbb{R}^2$ to $\mathbb{R}$)

and

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v}$$

<u>Example</u> $z = f(x,y) = e^{x^2+3y}$, $x(u,v) = u+2v$, $y(u,v) = u-v$

$$\frac{\partial f}{\partial x} = e^{x^2+3y} \cdot 2x, \quad \frac{\partial f}{\partial y} = e^{x^2+3y} \cdot 3, \quad \frac{\partial x}{\partial u} = 1, \ \frac{\partial x}{\partial v} = 2, \ \frac{\partial y}{\partial u} = 1, \ \frac{\partial y}{\partial v} = -1$$

$$\frac{\partial z}{\partial u} = e^{x^2+3y} \cdot 2x \cdot 1 + e^{x^2+3y} \cdot 3 \cdot 1 \qquad \frac{\partial z}{\partial v}$$