MATH 10C: Calculus III (Lecture B00)

mathweb.ucsd.edu/~ynemish/teaching/10c

Today: Local minima/maxima

Next: Strang 4.7

Week 7:

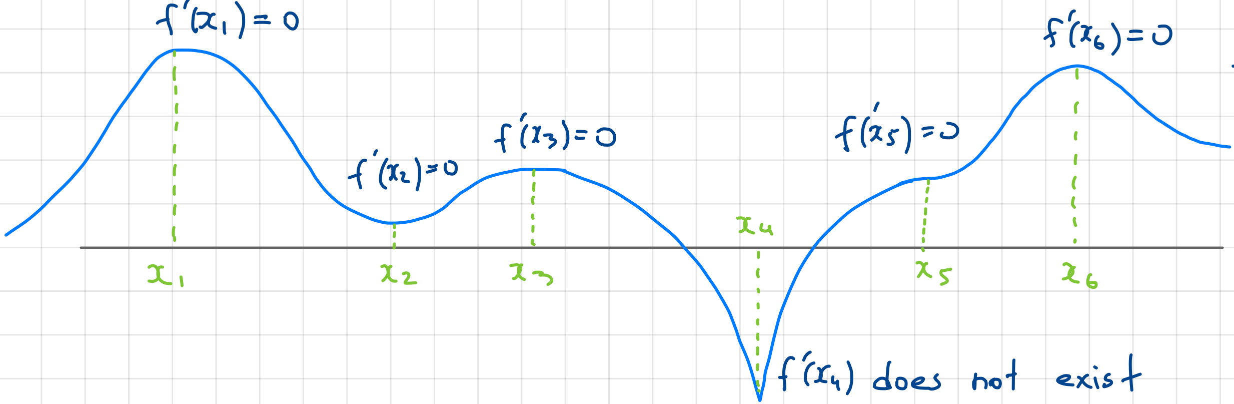
- homework 6 (due Friday, November 11)
- Midterm 2: Wednesday, November 16 (lectures 10-19)

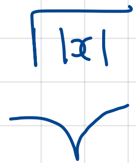
Maxima and minima of functions of one variable

Let $f : \mathbb{R} \to \mathbb{R}$ be a function of one variable.

The point $x_0 \in \mathbb{R}$ is called a critical point of f if either $f'(x_0) = 0$ or $f'(x_0)$ does not exist.

Any local maximum or local minimum of f is a critical point.





x_1, x_3, x_6 are points of local maximum, x_2, x_4 are points of local minimum; $x_1, x_2, x_3, x_4, x_5, x_6$ are critical points

Critical points of functions of two variables

Finding local minima/maxima in one dimension:

(i) identify critical points; (ii) determine which critical points are local minima/maxima.

We will extend this to functions of two variables. First, introduce the notion of a critical point for functions of two variables.

Def. Let $z = f(x,y)$ be a function of two variables defined at (x_0, y_0). Then (x_0, y_0) is called a critical point of f if either

- $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$ (i.e., $\nabla f(x_0, y_0) = \vec{0}$); or
- $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist

Critical points. Example

$$\frac{\partial}{\partial x} g(f(x,y)) = g'(f(x,y)) \cdot \frac{\partial}{\partial x} f(x,y)$$

Find the critical points of the function

$$f(x,y) = \sqrt{4y^2 - 9x^2 + 24y + 36x + 36}$$

Start by computing f_x and f_y and finding (x,y) s.t. $f_x(x,y)=0$ and $f_y(x,y)=0$ simultaneously

$$f_x(x,y) = \frac{-18x + 36}{2\sqrt{4y^2 - 9x^2 + 24y + 36x + 36}} = 0 \Rightarrow -18x + 36 = 0, \quad x = 2$$

$$f_y(x,y) = \frac{8y + 24}{2\sqrt{4y^2 - 9x^2 + 24y + 36x + 36}} = 0 \Rightarrow 8y + 24 = 0, \quad y = -3$$

Next, find all (x,y) for which f_x or f_y does not exist:

all x,y s.t. $4y^2 - 9x^2 + 24y + 36x + 36 = 0$, $\frac{(x-2)^2}{4} - \frac{(y+3)^2}{9} = 1$

Critical points. Example (cont.)

Therefore, $(2,-3)$ and $\left\{ (x,y) : \frac{(x-2)^2}{4} - \frac{(y+3)^2}{9} = 1 \right\}$ $(*)$

are possible critical points. We have to check that these points are in the domain of definition of f.

The domain of definition of f consists of all (x,y) s.t.

$$4y^2 - 9x^2 + 24y + 36x + 36 \geq 0$$

$$4(y^2 + 6y + 9) - 4\cdot 9 - 9(x^2 - 4x + 4) + 36 + 36 \geq 0$$

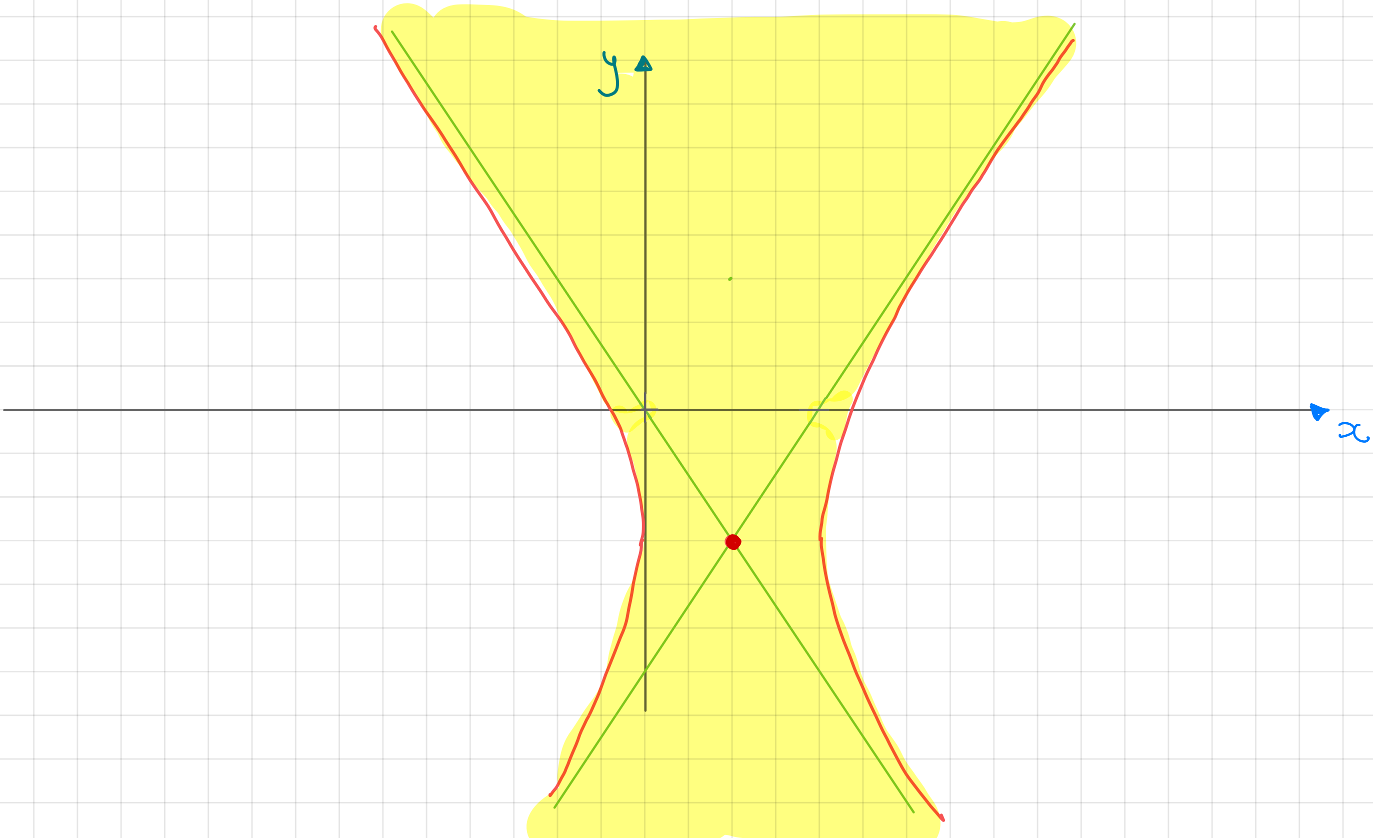
$$4(y+3)^2 - 9(x-2)^2 \geq -36, \quad \frac{(x-2)^2}{4} - \frac{(y+3)^2}{9} \leq 1 \quad (**)$$

Clearly, all points satisfying $(*)$ also satisfy $(**)$

Also, point $(2,-3)$ satisfies $(**)$. Therefore, the set of the critical points of f consists of $(2,-3)$ and all points of the hyperbola $\frac{(x-2)^2}{4} - \frac{(y+3)^2}{9} = 1$

Critical points. Example (cont.)

Here is the plot of the domain of f and the critical points of f



Local minimum/maximum

Def Let $z = f(x, y)$ be a function of two variables.
Then f has a local maximum at point (x_0, y_0) if

(*) $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) within some disk centered at (x_0, y_0). The number $f(x_0, y_0)$ is called the local maximum value. If (*) holds for all (x, y) in the domain of f, we say that f has a global maximum at (x_0, y_0)

Function f has a local minimum at point (x_0, y_0) if

(**) $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) within some disk centered at (x_0, y_0). The number $f(x_0, y_0)$ is called the local minimum value. If (**) holds for all (x, y) in the domain of f, we say that f has a global minimum at (x_0, y_0)

Local minima and local maxima are called local extrema.

Local extrema and critical points

Thm 4.16 Let $z = f(x,y)$ be a function of two variables, Suppose f_x and f_y each exist at (x_0,y_0). If f has a local extremum at (x_0,y_0), then (x_0,y_0) is a critical point of f (i.e., $\nabla f(x_0,y_0) = \vec{0}$)

Example At the very top of a mountain the ground is flat. If there was slope in some direction, then you could go higher. Similarly, at the lowest point of a crater the ground is also flat ($\nabla f = 0$).

But the fact that the ground is flat ($\nabla f(x_0,y_0) = 0$) does not necessarily mean that f has a local extremum at (x_0,y_0), it may be saddle point.