MATH 10C: Calculus III (Lecture B00)

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Today: Local minima/maxima

Next: Strang 4.7

Week 8:

- Midterm 2: Wednesday, November 16 (lectures 10-19)

Last time

<u>Def</u>. Let $z=f(x,y)$ be a function of two variables defined at (x_0,y_0). Then (x_0,y_0) is called a critical point of f if either

- $f_x(x_0,y_0)=0$, $f_y(x_0,y_0)=0$ (i.e. $\nabla f(x_0,y_0)=\vec{0}$); or
- $f_x(x_0,y_0)$ or $f_y(x_0,y_0)$ does not exist

Last time

Def Let $z = f(x, y)$ be a function of two variables.
Then f has a local maximum at point (x_0, y_0) if

(*) $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) within some disk centered at (x_0, y_0). The number $f(x_0, y_0)$ is called the local maximum value. If (*) holds for all (x, y) in the domain of f, we say that f has global maximum at (x_0, y_0).

Function f has a local minimum at point (x_0, y_0) if

(**) $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) within some disk centered at (x_0, y_0). The number $f(x_0, y_0)$ is called the local minimum value. If (**) holds for all (x, y) in the domain of f, we say that f has global minimum at (x_0, y_0).

Local minima and local maxima are called local extrema.

Last time

Thm 4.16 Let $z = f(x,y)$ be a function of two variables, Suppose f_x and f_y each exist at (x_0, y_0). If f has a local extremum at (x_0, y_0), then (x_0, y_0) is a critical point of f (i.e. $\nabla f(x_0, y_0) = 0$).

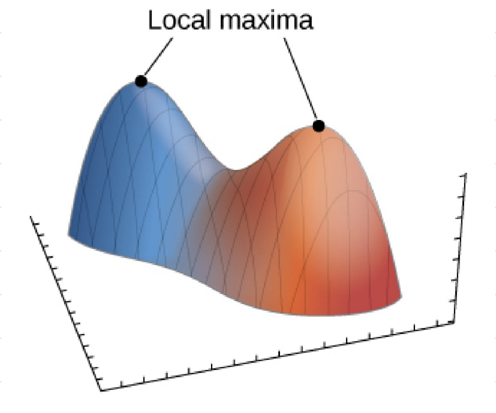
Example At the very top of a mountain the ground is flat. If there was slope in some direction, then you could go higher. Similarly, at the lowest point of a crater the ground is also flat ($\nabla f = 0$).

But the fact that the ground is flat ($\nabla f(x_0, y_0) = 0$) does not necessarily mean that f has a local extremum at (x_0, y_0), it may be a saddle point.

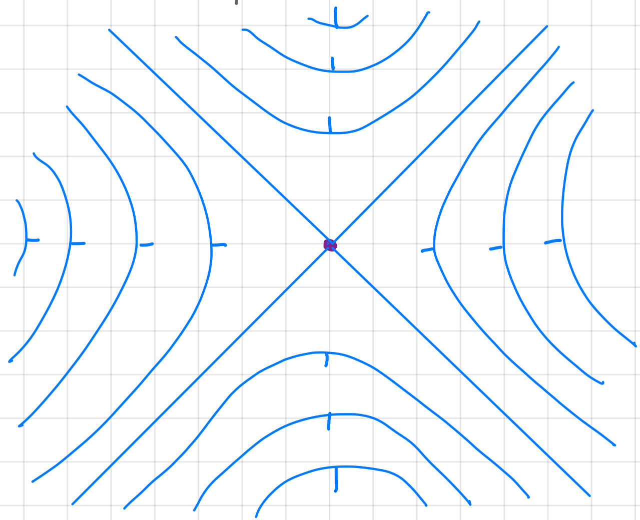
Saddle points

<u>Def.</u> Let $z=f(x,y)$ be a function of two variables.

We say that (x_0,y_0) is a saddle point if $f_x(x_0,y_0)=0$, $f_y(x_0,y_0)=0$, but f does not have a local extremum at (x_0,y_0)



Level curves around the saddle point have this shape



The second derivative test

Thm 4.17 (Second derivative test)

Suppose that $f(x,y)$ is a function of two variables for which the first- and second-order partial derivatives are continuous around (x_0,y_0). Suppose $f_x(x_0,y_0)=0$ and $f_y(x_0,y_0)=0$. Define

$$D = f_{xx}(x_0,y_0)f_{yy}(x_0,y_0) - \left(f_{xy}(x_0,y_0)\right)^2 = \begin{vmatrix} f_{xx}(x_0,y_0) & f_{xy}(x_0,y_0) \\ f_{xy}(x_0,y_0) & f_{yy}(x_0,y_0) \end{vmatrix}$$

(i) If $D>0$ and $f_{xx}(x_0,y_0)>0$, then f has a local minimum at (x_0,y_0)

(ii) If $D>0$ and $f_{xx}(x_0,y_0)<0$, then f has a local maximum at (x_0,y_0)

(iii) If $D<0$, then f has a saddle point at (x_0,y_0)

(iv) If $D=0$, then the test is inconclusive

Problem solving strategy

Problem:

Let $z = f(x,y)$ be a function of two variables for which the first- and second-ordered partial derivatives are continuous. Find local extrema.

Solution:

1. Determine critical points (x_0, y_0) where $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$
   Discard any points where f_x or f_y does not exist.
2. Caclulate D for each critical point
3. Apply the Second derivative test to determine if (x_0, y_0) is a local minimum, local maximum or a saddle point.

Local extrema. Examples

Find the critical points for the following function and use the second derivative test to find the local extrema

$$f(x,y) = x^3 + 2xy - 2x - 4y$$

Step 1: Compute ∇f and find the critical points

$$f_x = 3x^2 + 2y - 2$$

$$f_y = 2x - 4$$

f_x and f_y are defined and continuous everywhere

Find (x,y) such that $\nabla f(x,y) = \vec{0}$

$$\begin{cases} 3x^2 + 2y - 2 = 0, & 3 \cdot 2^2 + 2y - 2 = 0 \quad 2y = -10, \quad y = -5 \\ 2x - 4 = 0, & x = 2 \end{cases}$$

Function f has one critical point $(2,-5)$

Local extrema. Examples

Step 2: Compute D for $(2,-5)$

Start by computing $f_{xx}, f_{xy}, f_{yx}, f_{yy}$ at $(2,-5)$

$$f_{xx} = \frac{\partial}{\partial x} f_x = \frac{\partial}{\partial x}\left(3x^2+2y-2\right) = 6x \quad , \quad f_{xx}(2,-5) = 12$$

$$f_{xy} = \frac{\partial}{\partial y} f_x = \frac{\partial}{\partial y}\left(3x^2+2y-2\right) = 2 \quad , \quad f_{xy}(2,-5) = 2$$

$$f_{yy} = \frac{\partial}{\partial y} f_y = \frac{\partial}{\partial y}\left(2x-4\right) = 0 \quad , \quad f_{yy}(2,-5) = 0$$

$$D = \begin{vmatrix} 12 & 2 \\ 2 & 0 \end{vmatrix} = 12 \cdot 0 - 2 \cdot 2 = -4 < 0$$

Step 3: Second derivative test. $D<0$, so $(2,-5)$ is a saddle point

Local extrema. Examples

Find the critical points for the following function and use the second derivative test to find the local extrema

$$f(x,y) = xy\, e^{-\frac{x^2+y^2}{2}}$$

Step 1

$$f_x = y e^{-\frac{x^2+y^2}{2}} + xy e^{-\frac{x^2+y^2}{2}}(-x) = y(1-x^2)e^{-\frac{x^2+y^2}{2}}$$

$$f_y = x(1-y^2)e^{-\frac{x^2+y^2}{2}}$$

f_x and f_y are defined for all (x,y)

$$\begin{cases} f_x(x,y) = 0 \\ f_y(x,y) = 0 \end{cases} \Leftrightarrow \begin{cases} y(1-x^2) = 0 & \rightarrow \quad y=0,\ x=1,\ x=-1 \\ x(1-y^2) = 0 & \rightarrow \quad x=0,\ y=1,\ y=-1 \end{cases}$$

Critical points: $(0,0)$, $(1,1)$, $(1,-1)$, $(-1,1)$, $(-1,-1)$

Local extrema. Examples

Step 2 Second order partial derivatives

$$f_{xx} = \frac{\partial}{\partial x}\left[y(1-x^2)e^{-\frac{x^2+y^2}{2}} \right] = \frac{\partial}{\partial x}\left[y(1-x^2) \right] e^{-\frac{x^2+y^2}{2}} + y(1-x^2)\frac{\partial}{\partial x}\left[e^{-\frac{x^2+y^2}{2}} \right]$$

$$= y(-2x)e^{-\frac{x^2+y^2}{2}} + y(1-x^2)e^{-\frac{x^2+y^2}{2}}(-x)$$

$$= -xy\,e^{-\frac{x^2+y^2}{2}}(3-x^2)$$

$$f_{yy} = -xy\,e^{-\frac{x^2+y^2}{2}}(3-y^2)$$

$$f_{xy} = (1-x^2)(1-y^2)e^{-\frac{x^2+y^2}{2}}$$

Local extrema. Examples

$$f_{xx} = -xy(3-x^2)e^{-\frac{x^2+y^2}{2}}$$

$$f_{xy} = (1-x^2)(1-y^2)e^{-\frac{x^2+y^2}{2}}$$

$$f_{yy} = -xy(3-y^2)e^{-\frac{x^2+y^2}{2}}$$

Consider the critical point $(1,1)$

$$f_{xx}(1,1) = -1\cdot 1\cdot(3-1^2)e^{-\frac{1^2+1^2}{2}} = -2e^{-1}$$

$$f_{xy}(1,1) = (1-1^2)(1-1^2)e^{-1} = 0$$

$$f_{yy}(1,1) = -2e^{-1}$$

$$D = \begin{vmatrix} -2e^{-1} & 0 \\ 0 & -2e^{-1} \end{vmatrix} = 4e^{-2} > 0$$

$f_{xx}(1,1) < 0$, $D > 0$, therefore, f has a local maximum at $(1,1)$

Local extrema. Examples

$$f_{xx} = -xy(3-x^2)e^{-\frac{x^2+y^2}{2}}$$

$$f_{xy} = (1-x^2)(1-y^2)e^{-\frac{x^2+y^2}{2}}$$

$$f_{yy} = -xy(3-y^2)e^{-\frac{x^2+y^2}{2}}$$

Consider the critical point $(1,-1)$

$$f_{xx}(1,-1) = -1\cdot(-1)(3-1^2)e^{-\frac{1^2+1^2}{2}} = 2e^{-1}$$

$$f_{xy}(1,-1) = 0$$

$$f_{yy}(1,-1) = 2e^{-1}$$

$$D = \begin{vmatrix} 2e^{-1} & 0 \\ 0 & 2e^{-1} \end{vmatrix} = 4e^{-1}$$

Check: $(-1,1)$ local minimum

$(-1,-1)$ local maximum

$(0,0)$ saddle point

$f_{xx}(1,-1) > 0$, $D>0$, so f has local minimum at $(1,-1)$