MATH 10C: Calculus III (Lecture B00)

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Today: Absolute minima/maxima

Next: Strang 4.8

Week 8:

- Homework 7 due Wednesday, November 23

Recall: local and global minima and maxima

Def Let $z = f(x, y)$ be a function of two variables.

Then f has a local maximum at point (x_0, y_0) if

$(*)$ $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) within some disk centered at (x_0, y_0). The number $f(x_0, y_0)$ is called the local maximum value. If $(*)$ holds for all (x, y) in the domain of f, we say that f has a global maximum at (x_0, y_0)

Function f has a local minimum at point (x_0, y_0) if

$(**)$ $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) within some disk centered at (x_0, y_0). The number $f(x_0, y_0)$ is called the local minimum value. If $(**)$ holds for all (x, y) in the domain of f, we say that f has a global minimum at (x_0, y_0)

Local minima and local maxima are called local extrema.

Absolute (global) maxima and minima

Finding global minima/maxima for functions of one variable on a closed interval:

- find critical points
- check the critical values
- evaluate function at the endpoints of the interval.

We generalize this strategy to fuctions of two (or more) variables defined on a closed bounded set.

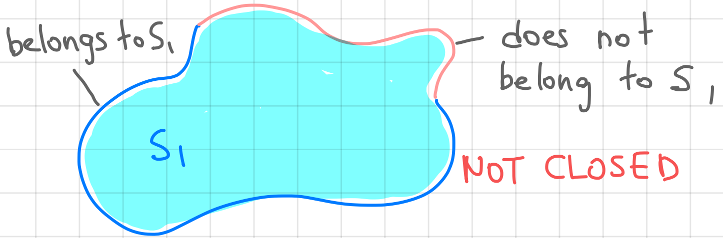
A set is called bounded if all the points of the set are contained in a disk (ball) of finite radius

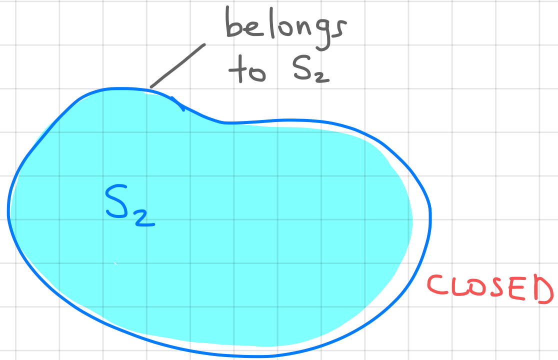
A set is called closed if it contains all its boundary points

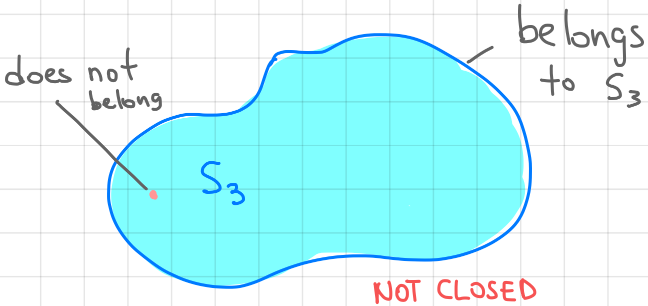
(A point P_0 is called a boundary point of a set S if any disk around P_0 contains points both inside and outside S)

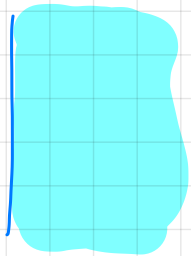
Extreme value theorem

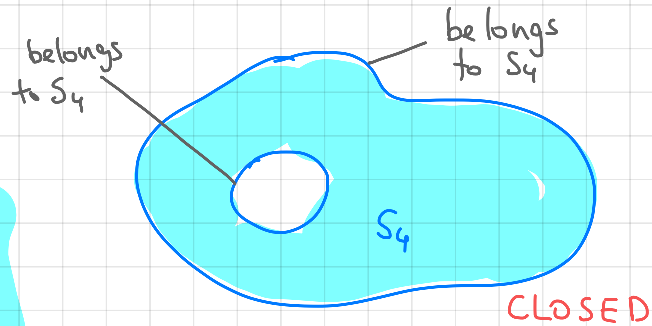
Closed sets ?











Theorem. A continuous function f on a closed bounded set D attains an absolute maximum value at some point in D and an absolute minimum value at some point in D.

Finding absolute minima and maxima

Thm. Assume $z = f(x,y)$ is a differentiable function of two variables defined on a closed bounded set D. Then f will attain the absolute maximum value and the absolute minimum value, which are, respectively, the largest and smallest values found among the following

(i) The values of f at the critical points in D

(ii) The values of f on the bounbary of D

Problem solving strategy for finding absolute max and min:

1. Determine the critical points of f in D
2. Calculate f at each of these critical points
3. Detetermine the max and min values of f on the boundary
4. Choose max/min from the values obtained in steps 2 and 3

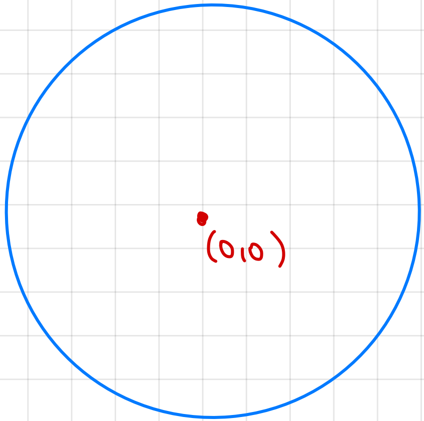
Example

Some villagers want to set up a communication tower within 1 km of their village. They want to put it at the highest elevation possible.

Suppose that the landscape around the village is described by $f(x,y) = x(x^2+2y^2-1)$ with the village center at $(0,0)$. What is the highest point within a (horizontal) distance of 1 km from $(0,0)$?

In other words, we have to maximize $f(x,y) = x(x^2+2y^2-1)$ on the set of all (x,y) with $x^2+y^2 \leq 1$

Example



The set $D = \{(x,y) \mid x^2+y^2 \le 1\}$ is a unit disk (including the boundary). It is a closed and bounded set.

Its boundary is a unit circle.

The maximum value can be inside the disk or on the boundary.

Remark: In general, finding the max/min value on the boundary may be nontrivial. One can parametrize the boundary as a curve in $\mathbb{R}^2$, and find the max/min of $f(x(t), y(t))$, where $(x(t), y(t))$ is the parametrization of the boundary, e.g., $(x(t), y(t)) = (\cos(t), \sin(t))$ $0 \le t \le 2\pi$

Example

Step 1: Determine the critical points inside D

$$f(x,y) = x(x^2+2y^2-1)$$

$$f_x = 1\cdot(x^2+2y^2-1) + x\cdot 2x = 3x^2+2y^2-1$$

$$f_y = 4xy$$

Find the critical points by solving

$$\begin{cases} 3x^2+2y^2-1 = 0 \\ 4xy = 0 \quad \longrightarrow \quad x=0 \quad \text{or} \quad y=0 \end{cases}$$

If $x=0$, $2y^2-1=0$, $y^2=\frac{1}{2}$, $y=\pm\frac{1}{\sqrt{2}}$

If $y=0$, $3x^2-1=0$, $x^2=\frac{1}{3}$, $x=\pm\frac{1}{\sqrt{3}}$

Four possible critical points: $\left(0,\frac{1}{\sqrt{2}}\right), \left(0,-\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{3}},0\right), \left(-\frac{1}{\sqrt{3}},0\right)$

Example

Check which points are in $D = \{(x,y) \mid x^2+y^2 \le 1\}$

$\left(0, \frac{1}{\sqrt{2}}\right), \left(0, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{3}}, 0\right), \left(-\frac{1}{\sqrt{3}}, 0\right)$

$$0^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 0 + \frac{1}{2} = \frac{1}{2} \le 1, \quad 0^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} \le 1, \quad \left(\frac{1}{\sqrt{3}}\right)^2 + 0^2 \le 1, \quad \left(-\frac{1}{\sqrt{3}}\right)^2 + 0^2 \le 1$$

All four points are inside the unit disk D

Step 2: Evaluate f at critical points

$$f(x,y) = x(x^2 + 2y^2 - 1)$$

$$f\left(0, \frac{1}{\sqrt{2}}\right) = 0 \cdot \left(0^2 + 2 \cdot \left(\frac{1}{\sqrt{2}}\right)^2 - 1\right) = 0 \qquad f\left(0, -\frac{1}{\sqrt{2}}\right) = 0$$

$$f\left(\frac{1}{\sqrt{3}}, 0\right) = \frac{1}{\sqrt{3}}\left(\left(\frac{1}{\sqrt{3}}\right)^2 + 2 \cdot 0 - 1\right) = \frac{1}{\sqrt{3}}\left(\frac{1}{3} - 1\right) = -\frac{2}{3\sqrt{3}}$$

$$f\left(-\frac{1}{\sqrt{3}}, 0\right) = -\frac{1}{\sqrt{3}}\left(\frac{1}{3} - 1\right) = \frac{2}{3\sqrt{3}}$$

Example

Step 3: Find the max value on the boundary

Parametrize the boundary

$$x(t) = \cos(t), \quad y(t) = \sin(t), \qquad 0 \le t \le 2\pi$$

Each point on the boundary (unit circle) can be written as $(\cos(t), \sin(t))$ for some $0 \le t \le 2\pi$

so the values of f on the boundary are

$$\begin{aligned} f(\cos(t), \sin(t)) &= \cos(t)\left(\cos^2(t) + 2\sin^2(t) - 1\right) \\ &= \cos(t)\left(\underbrace{\cos^2(t) + \sin^2(t)}_{=1} + \sin^2(t) \cancel{-1}\right) \\ &= \cos t \sin^2(t) \end{aligned}$$

We have to find the maximum of $\cos(t)\sin^2(t)$ for $0 \le t \le 2\pi$

Example

Maximizing $\cos(t)\sin^2(t)$:

Find the critical points

$$(\cos(t)\sin^2(t))' = -\sin(t)\sin^2(t) + \cos(t)\,2\sin(t)\cos(t)$$
$$= \sin(t)\left(2\cos^2(t) - \sin^2(t)\right)$$
$$= \sin(t)\left(3\cos^2(t) - 1\right) = 0$$

Critical points are points that satisfy

$y(t) = \sin(t) = 0$ or $3\cos^2(t) - 1 = 0$

$x^2 + y^2 = 1$

$x(t) = \cos^2(t) = \frac{1}{3}$, $x = \pm\frac{1}{\sqrt{3}}$

$x^2 = 1$, $x = \pm 1$

$y = \pm\sqrt{\frac{2}{3}}$

$(1,0)$, $(-1,0)$

$\left(\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}\right)$, $\left(-\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}\right)$, $\left(\frac{1}{\sqrt{3}}, -\sqrt{\frac{2}{3}}\right)$, $\left(-\frac{1}{\sqrt{3}}, -\sqrt{\frac{2}{3}}\right)$

Example

Compute the maximum on the boundary

$$f(x,y) = x(x^2+2y^2-1)$$

$f(1,0) = 0$ $\qquad$ $f(-1,0) = 0$

$f\left(\frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right) = \frac{2}{3\sqrt{3}}$

$f\left(-\frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right) = -\frac{2}{3\sqrt{3}}$

Step 4: Choose the absolute maximum

Inside the disk: $\left(-\frac{1}{\sqrt{3}}, 0\right)$, $f\left(-\frac{1}{\sqrt{3}}, 0\right) = \frac{2}{3\sqrt{3}}$

On the boundary: $f\left(\frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right) = \frac{2}{3\sqrt{3}}$

Conclusion: max value of f on D is $\frac{2}{3\sqrt{3}}$, achieved at $\left(-\frac{1}{\sqrt{3}}, 0\right)$ $\left(\frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right)$