MATH 10C: Calculus III (Lecture B00)

mathweb.ucsd.edu/~ynemish/teaching/10c

Today: Method of Lagrange multipliers
Next: Strang 4.8

Week 9:

- Homework 7 due Wednesday, November 23

Finding absolute minima and maxima

Thm. Assume $z = f(x,y)$ is a differentiable function of two variables defined on a closed bounded set D. Then f will attain the absolute maximum value and the absolute minimum value, which are, respectively, the largest and smallest values found among the following

(i) The values of f at the critical points of D

(ii) The values of f on the boundary of D

Problem solving strategy for finding absolute max and min:

1. Determine the critical points of f in D
2. Calculate f at each of these critical points
3. Detetermine the max and min values of f on the boundary
4. Choose max/min from the values obtained in steps 2 and 3

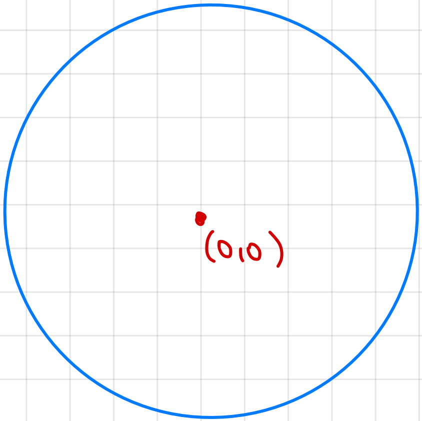
Example

Some villagers want to set up a communication tower within 1 km of their village. They want to put it at the highest elevation possible.

Suppose that the landscape around the village is described by $f(x,y) = x(x^2+2y^2-1)$ with the village center at $(0,0)$. What is the highest point within a (horizontal) distance of 1 km from $(0,0)$?

In other words, we have to maximize $f(x,y) = x(x^2+2y^2-1)$ on the set of all (x,y) with $x^2+y^2 \leq 1$

Example



The set $D = \{(x,y) \mid x^2+y^2 \leq 1\}$ is a unit disk (including the boundary). It is a closed and bounded set.

Its boundary is a unit circle.

The maximum value can be inside the disk or on the boundary.

Remark: In general, finding the max/min value on the boundary may be nontrivial. One can parametrize the boundary as a curve in $\mathbb{R}^2$, and find the max/min of $f(x(t), y(t))$, where $(x(t), y(t))$ is the parametrization of the boundary, e.g., $(x(t), y(t)) = (\cos(t), \sin(t))$, $t \in [0, 2\pi]$

Example

Compute the maximum on the boundary

$$f(x,y) = x(x^2+2y^2-1)$$

$f(1,0) = 0 \qquad f(-1,0) = 0$

$$f\left(\frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right) = \frac{1}{\sqrt{3}}\left(\frac{1}{3}+2\cdot\frac{2}{3}-1\right) = \frac{1}{\sqrt{3}}\left(\frac{5}{3}-1\right) = \frac{2}{3\sqrt{3}}$$

$$f\left(-\frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right) = -\frac{1}{\sqrt{3}}\left(\frac{1}{3}+2\cdot\frac{2}{3}-1\right) = -\frac{1}{\sqrt{3}}\cdot\frac{2}{3\sqrt{3}} = -\frac{2}{3\sqrt{3}}$$

Step 4: Choose the absolute maximum

Inside the disk: $f\left(-\frac{1}{\sqrt{3}}, 0\right) = \frac{2}{3\sqrt{3}}$

On the boundary: $f\left(\frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right) = \frac{2}{3\sqrt{3}}$

$\left(-\frac{1}{\sqrt{3}}, 0\right)$

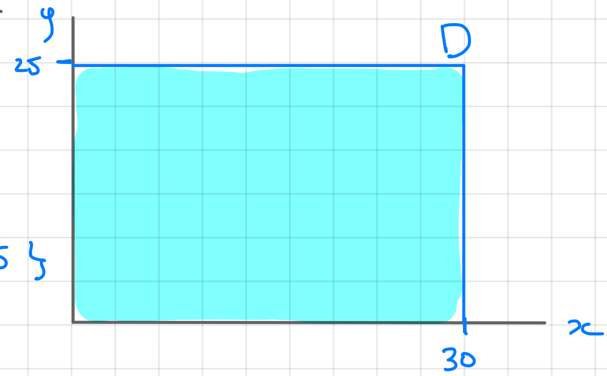
Conclusion: max value of f on D is $\frac{2}{3\sqrt{3}}$, attained at $\left(\frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right)$

Optimization problem with one constraint

Suppose you own a company that manufactures the golf balls. After analyzing the market you developed a model that describes your the company's profit as a function of the number x of golf balls sold and the number y of hours of advertizing

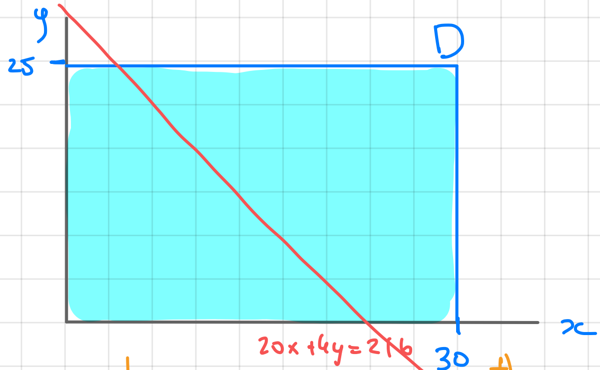
$$z = f(x,y) = 48x + 96y^2 - x^2 - 2xy - 9y^2$$

The maximum number of golf balls that can be produced is 30000, the max number of hours of advertizing is 25.



Maximize f on $D = \{(x,y) : 0 \le x \le 30,\ 0 \le y \le 25\}$

Optimization problem with one constraint



Maximize

$$z = f(x,y) = 48x + 96y^2 - x^2 - 2xy - 9y^2$$

on $D = \{(x,y) : 0 \le x \le 30000,\ 0 \le y \le 25\}$

Solution:

- find the critical points inside D and max among these points
- find the max on the boundary (parametrize the curve)

What if there is a budgetary constraint?

For example, what if we can only afford the combinations of x and y that satisfy $20x + 4y \le 216$? Now the boundary also includes (part of) the curve (line) $20x + 4y = 216$, and we have to maximize f on this boundary curve.

Optimization problem with one constraint

The constraints (like budgetary) are often of the form $g(x,y)=0$ for some function $g:\mathbb{R}^2\to\mathbb{R}$, and one has to maximize $f(x,y)$ on the curve defined by the equation $g(x,y)=0$. This is an example of an optimization problem with one constraint:

maximize $f(x,y)$ ← objective function

subject to the constraint $g(x,y)=0$

We can use the method described in the previous lecture: parametrize the curve, $\langle x(t),y(t)\rangle$, and find the critical points of $f(x(t),y(t))$ This can be simplified / shortened by using the method of Lagrange multipliers.

Method of Lagrange multipliers. One constraint

Problem: find the maximum/minimum of $f(x,y)$ on the curve C that is defined by the equation $g(x,y)=0$.

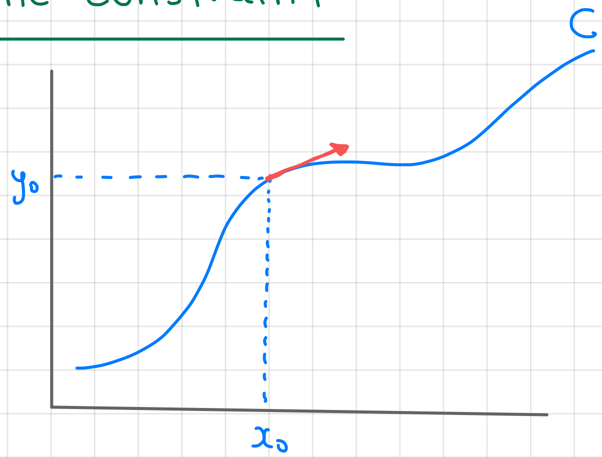
Suppose that (x_0, y_0) is the point of local max or min on C, and suppose that $\langle x(t), y(t) \rangle$ is a parametrization of C such that $x(0)=x_0$, $y(0)=y_0$. Then $t=0$ is a point of local max or min of $h(t) = f(x(t), y(t))$, which means that $h'(0)=0$. Now use the chain rule

$$h'(t) = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} = \frac{\partial f}{\partial x}(x(t), y(t))\, x'(t) + \frac{\partial f}{\partial y}(x(t), y(t)) \cdot y'(t)$$

$$h'(0) = \frac{\partial f}{\partial x}(x_0, y_0)\, x'(0) + \frac{\partial f}{\partial y}(x_0, y_0)\, y'(0) = \nabla f(x_0, y_0) \cdot \langle x'(0), y'(0) \rangle$$

$$= 0$$

Method of Lagrange multipliers. One constraint



$$\nabla f(x_0,y_0)\cdot\langle x'(0),y'(0)\rangle=0$$

$\langle x'(0),y'(0)\rangle$ is the tangent vector to the curve C at $t=0$

Conclusion 1: if (x_0,y_0) is a point of a local extremum of f on C, then $\nabla f(x_0,y_0)$ is orthogonal to the tangent vector of C at (x_0,y_0)

We also know that along the curve C $g(x(t),y(t))=0$,

so $\frac{\partial g}{\partial x}\cdot x'(t)+\frac{\partial g}{\partial y}\cdot y'(t)=0$ for any t (in particular $t=0$).

Thus (take $t=0$) $\frac{\partial g}{\partial x}(x_0,y_0)\,x'(0)+\frac{\partial g}{\partial y}(x_0,y_0)\,y'(0)=0$, i.e. $\nabla g(x_0,y_0)\cdot\langle x'(0),y'(0)\rangle=0$

Method of Lagrange multipliers. One constraint

Conclusion 2: If (x_0, y_0) is a local extremum of f on C, then both $\nabla f(x_0, y_0)$ and $\nabla g(x_0, y_0)$ are orthogonal to $\langle x'(0), y'(0) \rangle$.

Main conclusion: $\nabla f(x_0, y_0)$ and $\nabla g(x_0, y_0)$ are parallel

<u>Thm</u> (Method of Lagrange multipliers. One constraint)
Let f and g be functions of two variables with continuous partial derivatives at every point of some open set containing the smooth curve $g(x,y)=0$. Suppose that f, when restricted to the curve $g(x,y)=0$, has a local extremum at (x_0, y_0) and $\nabla g(x_0, y_0) \neq 0$. Then there is a number λ called Lagrange multiplier, for which

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

Method of Lagrange multipliers. One constraint

Problem: find the maximum/minimum of $f(x,y)$ on the curve C that is defined by the equation $g(x,y)=0$. Suppose that f is differentiable and C is smooth.

Problem solving strategy:

2. Set up the system of equations using the following template

$$\begin{cases} \nabla f(x_0,y_0) = \lambda \nabla g(x_0,y_0) \\ g(x_0,y_0) = 0 \end{cases}$$

3. Solve for x_0 and y_0 (may have multiple solutions)

4. The largest of the values of f at points (x_0,y_0) found above maximizes f on C; the smallest of the values minimizes f on C.

Method of Lagrange multipliers

Example Use the method of Largange multipliers to find the minimum value of $f(x,y) = x^2 + 4y^2 - 2x + 8y$ subject to the constraint $x + 2y = 7$.

1. Determine the objective function and the constraint function

$$f(x,y) = x^2 + 4y^2 - 2x + 8y \quad , \quad g(x,y) = x + 2y - 7$$

2. Set up the system of equations

$$f_x = 2x - 2 \ , \quad f_y = 8y + 8 \ , \quad g_x = 1 \ , \quad g_y = 2$$

$$\begin{cases} \langle 2x-2 , 8y+8 \rangle = \lambda \langle 1, 2 \rangle \\ x + 2y = 7 \end{cases} \qquad \begin{cases} 2x - 2 = \lambda \\ 8y + 8 = 2\lambda \\ x + 2y = 7 \end{cases}$$

Method of Lagrange multipliers

Example (cont.)

3. Solve the system of equations

$$\begin{cases} 2x-2 = \lambda & (1) \\ 8y+8 = 2\lambda & (2) \\ x+2y = 7 & (3) \end{cases}$$

Combine (1) and (2)

$\lambda = 2x-2 = 4y+4 = \lambda$

$\hookrightarrow x = 2y+3 \qquad (4)$

Plug (4) into (3):

$$2y+3+2y = 7$$
$$4y = 4$$
$$y = 1$$

Plug $y=1$ back into (4), $x = 2\cdot 1+3 = 5$.

The point $(5,1)$ is the only solution.

4. Evaluate f at $(5,1)$: $f(5,1) = 5^2+4\cdot 1^2 - 2\cdot 5 + 8\cdot 1 = 27$ Min or max?

Take any other point on the curve: $f(7,0) = 7^2+0-14+8 = 35$

Method of Lagrange multipliers

Example Maximize $f(x,y) = x(x^2+2y^2-1)$ subject to $x^2+y^2=1$.

1.

2.

3.

4. :

Method of Lagrange multipliers. Cobb-Douglas function

Company's production level is given by the Cobb-Douglas formula $f(x,y) = 2.5x^{0.45}y^{0.55}$, where x is the total number of labor hours, and y represents the total capital input. Suppose 1 unit of labor costs 40\$, one unit of capital costs 50\$. Use the Lagrange multipliers method to find the max value of $f(x,y) = 2.5x^{0.45}y^{0.55}$ subject to budgetary constraint of 500000\$.

1.

2. Set up the system of equations:

Method of Lagrange multipliers. Cobb-Douglas function

3. Solve the system (*):

Method of Lagrange multipliers. Cobb-Douglas function

$$x = \frac{45000}{8} = 5625 \qquad y = \frac{44000}{8} = 5500$$

4. The candidate for the maximum is $(5625, 5500)$.

Is this a maximum or a minimum?

Consider the function $2.5x^{0.45}y^{0.55}$ on the budgetary constraint line $40x + 50y = 500000$.

f can only have either one max on this line or one min on this line. Compute the value of f at any other point, e.g. $x = 0$, $y = 10000$.

Therefore, the production is maximized with 5625 units of labor and 5500 units of capital.