MATH 10C: Calculus III (Lecture B00)

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Today: Method of Lagrange multipliers
Next: Review

Week 10:

- Homework 8 due Friday, December 2
- CAPES

Final: Monday, December 5, 11:30 AM - 2:30 PM

Method of Lagrange multipliers. One constraint

Problem: find the maximum/minimum of $f(x,y)$ on the curve C that is defined by the equation $g(x,y)=0$. Suppose that f is differentiable and C is smooth.

Problem solving strategy:

2. Set up the system of equations using the following template

$$\begin{cases} \nabla f(x_0,y_0) = \lambda \nabla g(x_0,y_0) \\ g(x_0,y_0) = 0 \end{cases}$$

3. Solve for x_0 and y_0 (may have multiple solutions)

4. The largest of the values of f at points (x_0,y_0) found above maximizes f on C; the smallest of the values minimizes f on C.

More about step 4

Lagrange multipliers are used to find the critical points.

The points of local minima/maxima are critical points,

but critical points are not necessarily local minima/maxima

Suppose $(x_0, y_0), \dots, (x_n, y_n)$ are the points that satisfy the Lagrange multipliers equation and $f(x_0,y_0) \le f(x_1,y_1) \le \cdots \le f(x_n,y_n)$

- if $g(x,y)=0$ is bounded, then (x_0,y_0) minimizes f on $g(x,y)=0$, (x_n,y_n) maximizes f on $g(x,y)=0$ (we know max/min exist)
- if $g(x,y)=0$ is unbounded, visualize and determine whether f gets larger or smaller as (x,y) goes to infinity along $g(x,y)=0$
- if $g(x,y)=0$ is unbounded but we consider only a bounded part D of it, then check the value of f at the endpoints (boundary) of D

Lagrange multipliers in $\mathbb{R}^3$. Two constraints

Problem: maximize/minimize $f(x,y,z)$

subject to $g(x,y,z)=0$

$h(x,y,z)=0$

Problem solving strategy:

1. Determine the objective function f and the constraint functions g and h
2. Set up the system of equations

$$\nabla f(x_0,y_0,z_0) = \lambda_1 \nabla g(x_0,y_0,z_0) + \lambda_2 \nabla h(x_0,y_0,z_0)$$

$$g(x_0,y_0,z_0)=0 \quad , \quad h(x_0,y_0,z_0)=0$$

3. Solve the system for x_0, y_0, z_0 (may have multiple solutions)
4. Determine which of the points is max/min (if exists)

Lagrange multipliers in $\mathbb{R}^3$. Two constraints

Example Find the closest point to the origin on the line on intersection of the planes $2x+y+2z=9$, $5x+5y+7z=29$

Find the minimum of $f(x,y,z) = x^2+y^2+z^2$

subject to $2x+y+2z=9$

$5x+5y+7z=29$

1. $f(x,y,z) = x^2+y^2+z^2$, $g(x,y,z) = 2x+y+2z-9$, $h(x,y,z) = 5x+5y+7z-29$

2. Set up the system of equations:

$f_x = 2x$, $f_y = 2y$, $f_z = 2z$

$g_x = 2$, $g_y = 1$, $g_z = 2$

$h_x = 5$, $h_y = 5$, $h_z = 7$

$$\begin{cases} 2x = \lambda_1 \cdot 2 + \lambda_2 \cdot 5 \\ 2y = \lambda_1 \cdot 1 + \lambda_2 \cdot 5 \\ 2z = \lambda_1 \cdot 2 + \lambda_2 \cdot 7 \\ 2x+y+2z = 9 \\ 5x+5y+7z = 29 \end{cases}$$

Lagrange multipliers in $\mathbb{R}^3$. Two constraints

3. $$\begin{cases} 2x = \lambda_1 \cdot 2 + \lambda_2 \cdot 5 & (1) \\ 2y = \lambda_1 \cdot 1 + \lambda_2 \cdot 5 & (2) \\ 2z = \lambda_1 \cdot 2 + \lambda_2 \cdot 7 & (3) \\ 2x + y + 2z = 9 & (4) \quad (\times 2) \quad 2 \cdot 2x + 2y + 2 \cdot 2z = 18 \\ 5x + 5y + 7z = 29 & (5) \quad (\times 2) \quad 5 \cdot 2x + 5 \cdot 2y + 7 \cdot 2z = 58 \end{cases}$$

$$\begin{cases} 2 \cdot (2\lambda_1 + 5\lambda_2) + (\lambda_1 + 5\lambda_2) + 2 \cdot (2\lambda_1 + 7\lambda_2) = 18 \\ 5(2\lambda_1 + 5\lambda_2) + 5(\lambda_1 + 5\lambda_2) + 7(2\lambda_1 + 7\lambda_2) = 58 \end{cases}$$

$$\begin{cases} 4\lambda_1 + 10\lambda_2 + \lambda_1 + 5\lambda_2 + 4\lambda_1 + 14\lambda_2 = 18 \\ 10\lambda_1 + 25\lambda_2 + 5\lambda_1 + 25\lambda_2 + 14\lambda_1 + 49\lambda_2 = 58 \end{cases}$$

$$\begin{cases} 9\lambda_1 + 29\lambda_2 = 18 \\ 29\lambda_1 + 99\lambda_2 = 58 \end{cases}$$

$20\lambda_1 + 70\lambda_2 = 40$, $\lambda_1 = 2 - \frac{7}{2}\lambda_2$

$29\left(2 - \frac{7}{2}\lambda_2\right) + 99\lambda_2 = 58$, $58 + \left(-\frac{7 \cdot 29}{2} + 99\right)\lambda_2 = 58$

$\lambda_1 = 2$

$\lambda_2 = 0$

$$\boxed{\begin{aligned} x &= 2 \\ y &= 1 \\ z &= 2 \end{aligned}}$$

Lagrange multipliers in $\mathbb{R}^3$. Two constraints

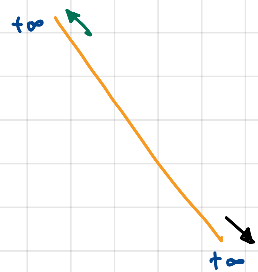
4. Min? Max?

$$f(2,1,2) = 2^2 + 1^2 + 2^2 = 9$$

Is the set determined by $2x + y + 2z = 9$ and $5x + 5y + 7z = 29$ bounded? Planes are not parallel, their intersection is a line

How does $f(x,y,z) = x^2 + y^2 + z^2$ behave as (x,y,z) tends to infinity along the line?

If (x,y,z) tends to infinity (in any direction), along the line, then $f(x,y,z)$ tends to $+\infty$



Conclusion: $(2,1,2)$ is the point of (absolute) minimum

Name (last, first): ______________________

Student ID: ______________________

☐ **Write your name and PID on the top of EVERY PAGE.**

☐ **The exam consists of 16 questions. Each question has only one correct answer. Be sure to completely fill in the appropriate bubble in the bubble answer sheet.**

☐ **DO NOT REMOVE ANY OF THE PAGES.**

☐ **No calculators, phones, or other electronic devices are allowed.**

☐ **You are allowed to use one 8.5 by 11 inch sheet of paper with hand-written notes (on both sides); no other notes (or books) are allowed.**

Name	**Version** Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ
ID	**Other**
Section	**Marking Instructions** Be sure to completely fill in the appropriate bubble. **Example** ● Ⓑ Ⓒ Ⓓ Ⓔ
Date	

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9	A	B	C	D	E	**34**	A	B	C	D	E	**59**	A	B	C	D	E	**84**	A	B	C	D	E
10	A	B	C	D	E	**35**	A	B	C	D	E	**60**	A	B	C	D	E	**85**	A	B	C	D	E
11	A	B	C	D	E	**36**	A	B	C	D	E	**61**	A	B	C	D	E	**86**	A	B	C	D	E
12	A	B	C	D	E	**37**	A	B	C	D	E	**62**	A	B	C	D	E	**87**	A	B	C	D	E
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