# MATH 10C: Calculus III (Lecture B00)

mathweb.ucsd.edu/~ynemish/teaching/10c

## Today: The cross product

## Next: Strang 2.~~4~~ 5

Week 2:

- homework 2 (due Monday, October 10)
- survey on Canvas Quizzes (due Friday, October 7)

# The cross product

**Def** Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$.

Then the cross product of $\vec{u}$ and $\vec{v}$ is vector

$$\vec{u} \times \vec{v} = (u_2 v_3 - u_3 v_2)\vec{i} - (u_1 v_3 - u_3 v_1)\vec{j} + (u_1 v_2 - u_2 v_1)\vec{k}$$

$$= \langle u_2 v_3 - u_3 v_2 \,,\, -(u_1 v_3 - u_3 v_1) \,,\, u_1 v_2 - u_2 v_1 \rangle$$

**Example**

$$\vec{p} = \langle 1, 2, 3 \rangle, \quad \vec{q} = \langle -1, 2, 0 \rangle$$

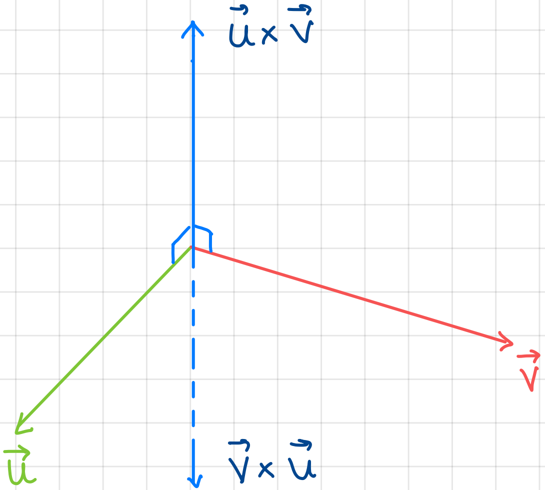
$$\vec{p} \times \vec{q} = \langle 2\cdot 0 - 3\cdot 2 \,,\, -(1\cdot 0 - 3\cdot(-1)) \,,\, 1\cdot 2 - 2\cdot(-1) \rangle$$

$$= \langle -6, -3, 4 \rangle$$

$$\vec{p}\cdot(\vec{p}\times\vec{q}) = \langle 1,2,3\rangle \cdot \langle -6,-3,4\rangle = 0 \quad , \quad \vec{q}\cdot(\vec{p}\times\vec{q}) = \langle -1,2,0\rangle\cdot\langle -6,-3,4\rangle = 0$$

# The cross product

Fact: Vector $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$ !




and the direction is determined by the right-hand rule.

Indeed, $\vec{u} \times \vec{v} = - \vec{v} \times \vec{u}$

(anticommutative)

$\vec{p} = \langle 1, 2, 3 \rangle$, $\vec{q} = \langle -1, 2, 0 \rangle$, $\vec{p} \times \vec{q} = \langle -6, -3, 4 \rangle$

$$\vec{q} \times \vec{p} = \langle 2 \cdot 3 - 0 \cdot 2, -((-1) \cdot 3 - 0 \cdot 1), (-1) \cdot 2 - 2 \cdot 1 \rangle$$

$$= \langle 6, 3, -4 \rangle = - \vec{p} \times \vec{q}$$

# Properties of the cross product

**Exercise** $\vec{i} \times \vec{j} = \langle 1,0,0\rangle \times \langle 0,1,0\rangle = \langle 0,0,1\rangle = \vec{k}$

$$\vec{i} \times \vec{j} = \vec{k} \qquad \vec{j} \times \vec{k} = \vec{i} \qquad \vec{k} \times \vec{i} = \vec{j}$$

**Theorem 2.6** Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^3$. Then

(i) $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$

(ii) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

(iii) $c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v})$

(iv) $\vec{u} \times \vec{0} = \vec{0} \times \vec{u} = \vec{0}$

(v) $\vec{v} \times \vec{v} = \vec{0}$

(vi) $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$

For proof expand both sides in terms of components of $\vec{u}, \vec{v}, \vec{w}$

# Properties of cross product

In general, $(\vec{u}\times\vec{v})\times\vec{w} \neq \vec{u}\times(\vec{v}\times\vec{w})$

$$(\vec{i}\times\vec{i})\times\vec{j} = \vec{0}\times\vec{j} = \vec{0}$$

$$\vec{i}\times(\vec{i}\times\vec{j}) = \vec{i}\times\vec{k} = -\vec{k}\times\vec{i} = -\vec{j}$$

**Example** (a) Calculate $(2\vec{i})\cdot\left((3\vec{j})\times\vec{k} + \vec{i}\times(-4)\vec{k}\right)$

$$(2\vec{i})\times\left((3\vec{j})\times\vec{k} + \vec{i}\times(-4)\vec{k}\right) =$$

$$=$$

$$=$$

(b) Show that $\vec{u}\times\vec{v}$ is orthogonal to $\vec{u}$ and $\vec{v}$

$$\vec{u}\cdot(\vec{u}\times\vec{v}) \overset{(vi)}{=} (\vec{u}\times\vec{u})\cdot\vec{v} = \vec{0}\cdot\vec{v} = 0$$

$$\vec{v}\cdot(\vec{u}\times\vec{v}) = (\vec{u}\times\vec{v})\cdot\vec{v} = \vec{u}\cdot(\vec{v}\times\vec{v}) = \vec{u}\cdot\vec{0} = 0$$

# Magnitude of the cross product

Fact. Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^3$. Then

$$\|\vec{u}\|^2 \|\vec{v}\|^2 = \|\vec{u} \times \vec{v}\|^2 + (\vec{u} \cdot \vec{v})^2 \qquad (*)$$

Proof. Expand both sides using components $\vec{u} = \langle u_1, u_2, u_3 \rangle$
$\vec{v} = \langle v_1, v_2, v_3 \rangle$

Theorem 2.7 Let $\vec{u}$ and $\vec{v}$ be vectors, let $\theta$ be the angle between them. Then

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \sin\theta$$

Proof From $(*)$ $\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2$

From Thm. 2.4 $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$. Then

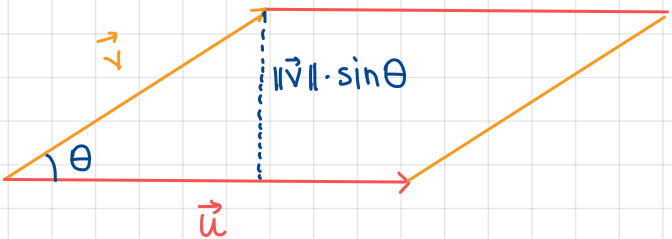
$$\|\vec{u} \times \vec{v}\| = \sqrt{\|\vec{u}\|^2 \|\vec{v}\|^2 - \|\vec{u}\|^2 \|\vec{v}\|^2 (\cos\theta)^2} = \|\vec{u}\| \|\vec{v}\| \underbrace{\sqrt{1 - (\cos\theta)^2}}_{= \sin\theta}$$

# Geometric interpretation

Summary: Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^3$.
Then $\vec{u} \times \vec{v}$ is a vector in $\mathbb{R}^3$ such that

- $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$ (right-hand rule)
- $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \sin\theta$ with $\theta$ = angle between $\vec{u}$ and $\vec{v}$

Consider a parallelogram spanned by vectors $\vec{u}$ and $\vec{v}$



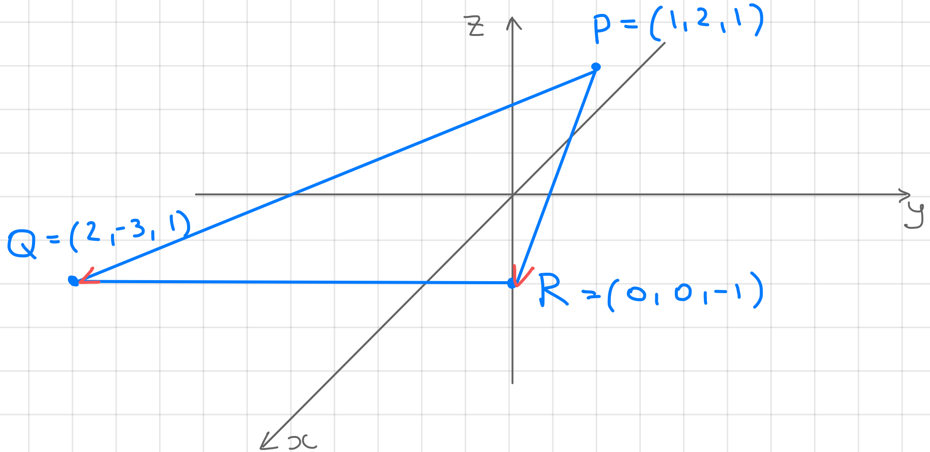

Area (  ) $= \|\vec{u}\| \cdot \|\vec{v}\| \cdot \sin\theta$

$= \|\vec{u} \times \vec{v}\|$

Conclusion: magnitude of $\vec{u} \times \vec{v}$ is equal to the area of the parallelogram spanned by $\vec{u}$ and $\vec{v}$

# Example

Let $P=(1,2,1)$, $Q=(2,-3,1)$, $R=(0,0,-1)$ be the vertices on a triangle. Find its area.




$\overrightarrow{PQ} = \langle 1, -5, 0 \rangle$ , $\overrightarrow{PR} = \langle -1, -2, -2 \rangle$ , $\text{Area}(\Delta) = \frac{1}{2} \| \overrightarrow{PQ} \times \overrightarrow{PR} \| = \frac{1}{2}\sqrt{153}$

$\overrightarrow{PQ} \times \overrightarrow{PR} = 10\vec{i} - (-2)\vec{j} + (-7)\vec{k} = \langle 10, 2, -7 \rangle$ , $\| \overrightarrow{PQ} \times \overrightarrow{PR} \| = \sqrt{10^2 + 2^2 + 7^2} = \sqrt{153}$