MATH 10C: Calculus III (Lecture B00)

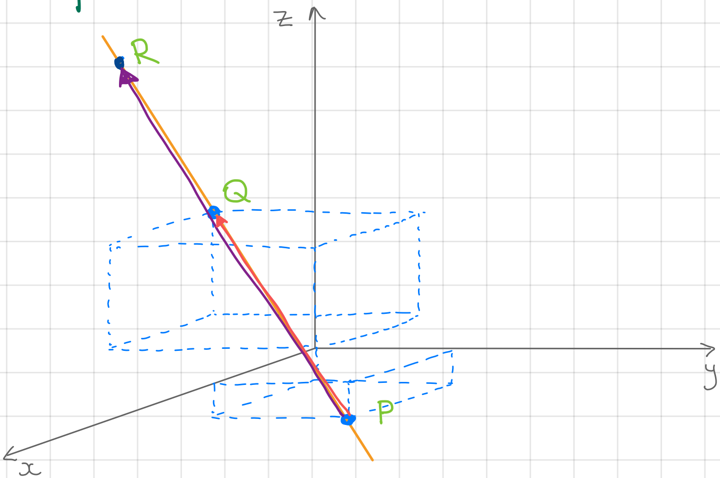
mathweb.ucsd.edu/~ynemish/teaching/10c

Today: Equations of lines and planes
Next: Strang 3.1

Week 3:

- homework 2 (due Monday, October 10)
- OH schedule updated

Equation for a line in space



To describe a line in $\mathbb{R}^3$ we must know either

(a) two points on the line,

or (b) one point and direction.

Thm 2.11 (Parametric and symmetric eqs. of a line)

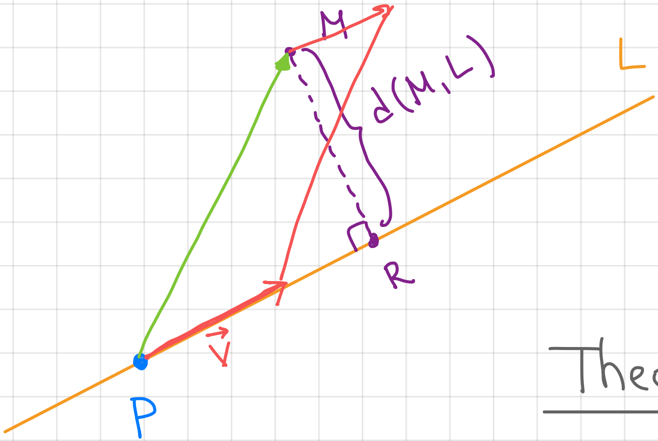
A line parallel to vector $\vec{v} = \langle a, b, c \rangle$ and passing through $P = (x_0, y_0, z_0)$ can be described by the following parametric equations: $x = x_0 + ta, \; y = y_0 + tb, \; z = z_0 + tc \;, \; t \in \mathbb{R}$

If a, b and c are all nonzero, L can be described by the symmetric equation $\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$

Distance between a point and a line

Consider the line L through point P with direction vector $\vec{v}$.

Suppose M is not on the line. What is the distance between L and M?



Area of the parallelogram spanned by $\vec{PM}$ and $\vec{v}$

$$\|\vec{PM} \times \vec{v}\| = \|\vec{v}\| \cdot d(M,L)$$

Theorem 2.12. Let L be a line passing through P with direction vector $\vec{v}$. If M is any point not on L, then

$$d(M,L) = \frac{\|\vec{v} \times \vec{PM}\|}{\|\vec{v}\|}$$

Distance between a point and a line

Example

Find the distance between $M=(3,2,1)$ and the line $\frac{x-5}{2} = \frac{y+2}{2} = -z$ | $\frac{x-5}{2} = \frac{y-(-2)}{2} = \frac{z-0}{-1}$

Identify a point on the line : $P=(5,-2,0)$

Identify the direction vector of the line : $\vec{v} = \langle 2,2,-1 \rangle$

Compute $\vec{PM} = \langle -2, 4, 1 \rangle$, $\vec{PM} \times \vec{v} = \langle -6, 0, -12 \rangle$

Finally, $\|\vec{v}\| = \sqrt{2^2+2^2+1^2} = 3$, $\|\vec{PM} \times \vec{v}\| = \sqrt{6^2+12^2} = \sqrt{36+144}$
$= \sqrt{180}$

$$d(M,L) = \frac{\sqrt{180}}{3} = \sqrt{20} = 2\sqrt{5}$$

Relationships between lines in $\mathbb{R}^3$

Let L_1 and L_2 be two lines in $\mathbb{R}^3$. Then the following four possibilities exist:

		L_1 and L_2 share a common point	
		YES	NO
Direction vectors of L_1 and L_2 are parallel	YES	Equal	Parallel but not equal
	NO	Intersecting	skew $\parallel$ not parallel not intersecting

Relationships between lines in $\mathbb{R}^3$

Example

L_1: direction vector $\vec{v}_1 = \langle 1, 2, 0 \rangle$, passing through $P_1 = (0,0,1)$

L_2: direction vector $\vec{v}_2 = \langle -3, -6, 0 \rangle$, passing through $P_2 = (1,2,3)$

L_3: direction vector $\vec{v}_3 = \langle 1, -1, 1 \rangle$, passing through $P_3 = (-1,4,-1)$

① L_1 and L_2 $\quad$ $\vec{v}_1$ is parallel to $\vec{v}_2$, $\vec{v}_1 = -\frac{1}{3}\vec{v}_2$, therefore,

L_1 and L_2 are either equal or parallel but not equal

Write equations for L_1:

$$\begin{cases} x = 0 + 1 \cdot t \\ y = 0 + 2 \cdot t \\ z = 1 \end{cases}$$

If L_1 and L_2 are equal, then $P_2 \in L_1$

$$\begin{cases} 1 = t \\ 2 = 2t \\ 3 = 1 \end{cases}$$

No solution, so L_1 and L_2 are parallel but not equal

Relationships between lines in $\mathbb{R}^3$

Example ② L_1 and L_3

(i) $\vec{v}_1 = \langle 1, 2, 0 \rangle$, $\vec{v}_3 = \langle 1, -1, 1 \rangle$. Are $\vec{v}_1$ and $\vec{v}_3$ parallel?

Parallel if and only if $\vec{v}_1 = k\vec{v}_3$ for some $k \in \mathbb{R}$

$$\begin{cases} 1 = k \\ 2 = -k \\ 0 = k \end{cases}$$

this system has no solutions, so direction vectors are L_1 and L_3 are

(ii) Do L_1 and L_3 have a point in common?

If $Q = (x, y, z)$ belongs to both L_1 and L_3, then the coordinates of Q must satisfy both equations

$$\begin{cases} x = t \\ y = 2t \\ z = 1 \end{cases} \quad \text{and} \quad \begin{cases} x = -1 + s \\ y = 4 - s \\ z = -1 + s \end{cases}$$

for some $s, t \in \mathbb{R}$

Relationships between lines in $\mathbb{R}^3$

$$\begin{cases} x = t, \\ y = 2t, \\ z = 1, \end{cases} \quad \text{and} \quad \begin{cases} x = -1 + s \\ y = 4 - s \\ z = -1 + s \end{cases} \quad \text{for some } s, t \in \mathbb{R}$$

Equate the right-hand sides of the above equations

$$\begin{cases} t = -1 + s \\ 2t = 4 - s \\ 1 = -1 + s \end{cases}$$

If this system has a solution then L_1 and L_3 intersect

From the last equation we have $s = 2$. Substituting $s = 2$ into the first two equations gives $t = -1 + 2 = 1$, $2t = 4 - 2 = 2$ | $t = 1$

$t = 1$: $\begin{array}{l} x = 1 \\ y = 2 \\ z = 1 \end{array}$ $\quad s = 2$: $\begin{array}{l} x = 1 \\ y = 2 \\ z = 1 \end{array}$ $\quad Q = (1, 2, 1)$

↳ L_1 and L_3 intersect at $Q = (1, 2, 1)$

Relationships between lines in $\mathbb{R}^3$ Exercise

Example ③: L_2 and L_3

L_2: direction vector $\vec{v}_2 = \langle -3, -6, 0 \rangle$, passing through $P_2 = (1, 2, 3)$

L_3: direction vector $\vec{v}_3 = \langle 1, -1, 1 \rangle$, passing through $P_3 = (-1, 4, -1)$

Since $\vec{v}_2$ and $\vec{v}_3$ are not parallel, L_2 and L_3 are either intersecting or skew. We have to check if L_2 and L_3 have a point in common.

L_2: $\{$ $\qquad$ L_3: $\{$ $\qquad$ Equate: $\{$

$\rightarrow \{ \qquad \} \qquad \{$

Relationships between lines in $\mathbb{R}^3$ Solution

Example ③ : L_2 and L_3

L_2: direction vector $\vec{v}_2 = \langle -3, -6, 0 \rangle$, passing through $P_2 = (1, 2, 3)$

L_3: direction vector $\vec{v}_3 = \langle 1, -1, 1 \rangle$, passing through $P_3 = (-1, 4, -1)$

Since $\vec{v}_2$ and $\vec{v}_3$ are not parallel, L_2 and L_3 are either intersecting or skew. We have to check if L_2 and L_3 have a point in common.

$$L_2: \begin{cases} x = 1 - 3t \\ y = 2 - 6t \\ z = 3 \end{cases} \quad L_3: \begin{cases} x = -1 + s \\ y = 4 - s \\ z = -1 + s \end{cases} \quad \text{Equate: } \begin{cases} 1 - 3t = -1 + s \\ 2 - 6t = 4 - s \\ 3 = -1 + s \end{cases}$$

$$s = 4 \rightarrow \begin{cases} 1 - 3t = -1 + 4 \\ 2 - 6t = 4 - 4 \end{cases} \quad \begin{cases} -3t = 2 \\ -6t = -2 \end{cases} \quad \begin{cases} t = -\frac{2}{3} \\ t = \frac{1}{3} \end{cases}$$

no solution

L_2 and L_3 are SKEW!