MATH 10C: Calculus III (Lecture B00)

mathweb.ucsd.edu/~ynemish/teaching/10c

Today: Equations of a plane

Next: Strang 3.1

Week 3:

Tue

- homework 3 (due ~~Monday~~, October ~~17~~ 18)
- Midterm 1: Wednesday, October 19 (vectors, dot product, cross product, equations of lines and planes)

Planes

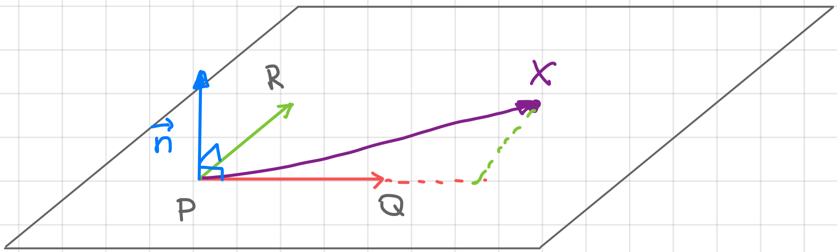
Two points determine a line: for any two points P, Q (in $\mathbb{R}^2$ or $\mathbb{R}^3$) there exists a unique line passing through P and Q. A point X is in the line through P and Q if $\vec{PX}$ is a multiple of $\vec{PQ}$, i.e., $\vec{PX} = t\,\vec{PQ}$ for some $t \in \mathbb{R}$.

Three points (that do not all lie on the same line) determine a plane: for any three points P, Q and R in $\mathbb{R}^3$ that do not all lie on the same line, there exists a unique plane that passes through these three points.

A point X is in the plane passing through P, Q and R if $\vec{PX}$ is a linear combination of vectors $\vec{PQ}$ and $\vec{PR}$

$$\vec{PX} = t\,\vec{PQ} + s\,\vec{PR} \quad \text{for some } t, s \in \mathbb{R}$$

Equation of a plane



Another way to describe a plane is by identifying a point in the plane and a vector that is perpendicular (orthogonal) to the plane . If P is a point in the plane and vector $\vec{n}$ is orthogonal to the plane (called the normal vector) then point X is in this plane if and only if

$$\vec{n} \perp \overrightarrow{PX}, \quad \vec{n} \cdot \overrightarrow{PX} = 0$$

(vector equation of a plane)

Equation of a plane

Consider a plane containing point $P = (x_0, y_0, z_0)$ with normal vector $\vec{n} = \langle a, b, c \rangle$. Then point $X = (x, y, z)$ belong to this plane if and only if

(*) $$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

scalar equation of a plane

If we denote $d := -ax_0 - by_0 - cz_0$, then (*) becomes

$$ax + by + cz + d = 0$$

general form of the equation of a plane

Suppose that we know the coordinates of three points P, Q, R in the plane. How can we find a normal vector to this plane?

$$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$$

Example

Write the vector equation for the plane containing points $P=(1,1,0)$, $Q=(-2,1,1)$, $R=(0,0,1)$

$$\vec{PQ} = \langle -3,0,1\rangle, \quad \vec{PR} = \langle -1,-1,1\rangle$$

Compute the normal vector to the plane

$$\vec{n} = \vec{PQ} \times \vec{PR} = \langle -3,0,1\rangle \times \langle -1,-1,1\rangle = \langle 1,2,3\rangle$$

Point $X=(x,y,z)$ is in the plane if

$\vec{n} \cdot \vec{PX} = 0$, or equivalently

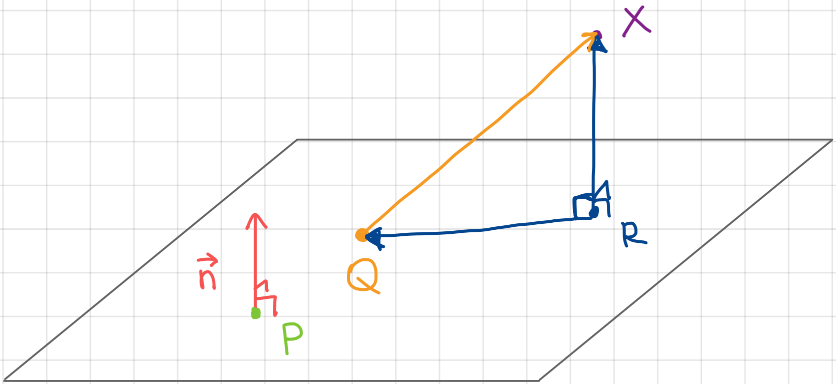
$\langle 1,2,3\rangle \cdot \langle x-1, y-1, z-0\rangle = 0$ — vector equation of the plane

equivalent

$1\cdot(x-1) + 2(y-1) + 3\cdot z = 0$ — scalar equation

$x + 2y + 3z - 3 = 0$ — general form

Distance between a plane and a point



$d = \|\vec{RX}\|$ such that

RX $\perp$ to the plane

($\|\vec{RX}\| < \|\vec{QX}\|$ for any Q in the plane)

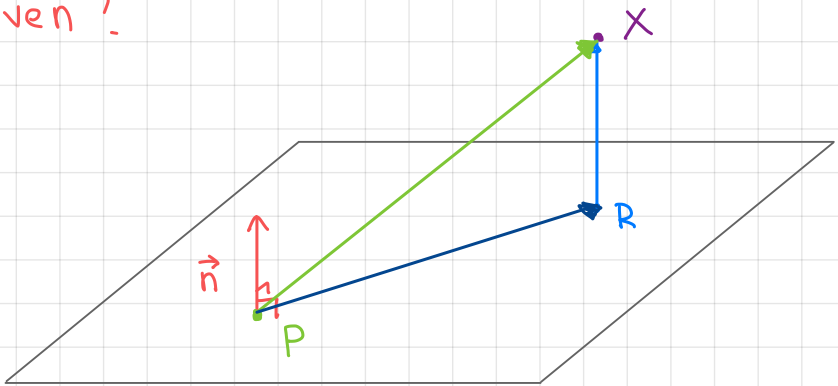
Consider a plane with point P and normal vector $\vec{n}$. Suppose that point X does not belong to this plane. The distance d between X and the plane is the smallest distance between X and points in the plane.

If $\vec{RX}$ is orthogonal to the plane (parallel to $\vec{n}$), then $\vec{RX} \perp \vec{RQ}$ for any point $Q \neq R$ in the plane, $\|\vec{RX}\| < \|\vec{QX}\|$

Distance between a plane and a point

Conclusion: $d = \|\overrightarrow{RX}\|$, where R is in the plane and $\overrightarrow{RX}$ is perpendicular to the plane ($\overrightarrow{RX}$ is parallel to $\vec{n}$)

How to find $\overrightarrow{RX}$ (and $\|\overrightarrow{RX}\|$) if P and $\vec{n}$ are given?



distance from point X to the plane with P and $\vec{n}$

$$\overrightarrow{RX} = \text{proj}_{\vec{n}} \overrightarrow{PX} = \frac{\overrightarrow{PX} \cdot \vec{n}}{\|\vec{n}\|^2}\vec{n}, \quad \|\overrightarrow{RX}\| = \frac{|\overrightarrow{PX} \cdot \vec{n}|}{\|\vec{n}\|^2} \cdot \|\vec{n}\| = \frac{|\overrightarrow{PX} \cdot \vec{n}|}{\|\vec{n}\|} = d$$

Distance between a plane and a point

Example

Find the distance between the point $X = (0,0,0)$ and the plane given by $x + 2y + 3z - 3 = 0$.

This is the equation in the general form. First find the normal vector $\vec{n} = \langle 1, 2, 3 \rangle$

Next we need a point in the plane (any point), i.e., any numbers x_0, y_0, z_0 such that $x_0 + 2y_0 + 3z_0 - 3 = 0$.

We can take $x_0 = 0$, $y_0 = 0$, which requires that $z_0 = 1$. $P = (0,0,1)$

$\vec{PX} = \langle 0, 0, -1 \rangle$ Then the distance from X to the plane is

$$d = \frac{|\langle 1,2,3 \rangle \cdot \langle 0,0,-1 \rangle|}{\|\langle 1,2,3 \rangle\|} = \frac{|-3|}{\sqrt{1^2+2^2+3^2}} = \frac{3}{\sqrt{14}}$$

Parallel and intersecting planes

Let P_1 and P_2 be two planes in $\mathbb{R}^3$. Then the following possibilities exist:

		P_1 and P_2 share a common point	
		YES	NO
Normal vectors of P_1 and P_2 are parallel	YES	Equal	Parallel but not equal
	NO	Intersecting	

If two planes intersect, the intersection is a line!