Name (last, first): ______________________________

Student ID: ______________________________

☐ **Write your name and PID on the top of EVERY PAGE.**

☐ **Write the solutions to each problem on separate pages. CLEARLY INDICATE on the top of each page the number of the corresponding problem. Different parts of the same problem can be written on the same page (for example, part (a) and part (b)).**

☐ **The exam consists of 4 questions. Your answers must be carefully justified to receive credit.**

☐ **This exam will be scanned. Make sure you write ALL SOLUTIONS on the paper provided. DO NOT REMOVE ANY OF THE PAGES.**

☐ **No calculators, phones, or other electronic devices are allowed.**

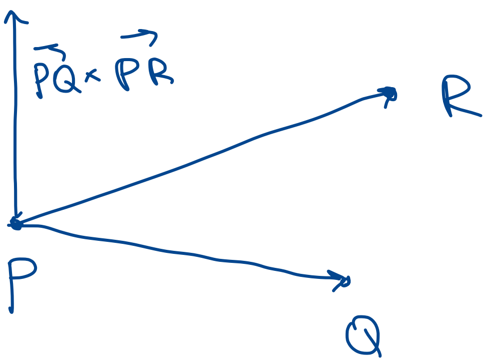
☐ **Remember this exam is graded by a human being. Write your solutions NEATLY AND COHERENTLY, or they risk not receiving full credit.**

☐ **You are allowed to use one 8.5 by 11 inch sheet of paper with handwritten notes (on both sides); no other notes (or books) are allowed.**

1. (20 points) Find *two* unit vectors orthogonal to the plane that contains the points

$$P = (0, 1, 2), \quad Q = (2, -1, 2), \quad R = (-1, 0, 1).$$

Also, find an equation for this plane.



$\vec{PQ} = \langle 2, -2, 0 \rangle$, $\vec{PR} = \langle -1, -1, -1 \rangle$

Vector $\vec{PQ} \times \vec{PR}$ is orthogonal to both $\vec{PQ}$ and $\vec{PR}$ and, therefore, orthogonal to the plane containing points P, Q, R

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & 0 \\ -1 & -1 & -1 \end{vmatrix} = \vec{i} \begin{vmatrix} -2 & 0 \\ -1 & -1 \end{vmatrix} - \vec{j} \begin{vmatrix} 2 & 0 \\ -1 & -1 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & -2 \\ -1 & -1 \end{vmatrix}$$

$$= \vec{i} \cdot 2 - \vec{j}(-2) + \vec{k}(-4) = \langle 2, 2, -4 \rangle$$

$$\vec{u}_1 = \frac{\langle 2, 2, -4 \rangle}{\|\langle 2, 2, -4 \rangle\|} = \frac{\langle 2, 2, -4 \rangle}{\sqrt{4+4+16}} = \frac{\langle 2, 2, -4 \rangle}{\sqrt{24}} = \left\langle \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}} \right\rangle$$

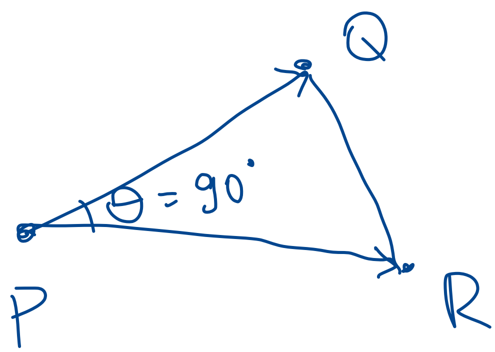
The second unit vector is obtained by multiplying $\vec{u}_1$ by -1: $\vec{u}_2 = \left\langle -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right\rangle$

Equation of the plane: $2 \cdot (x - 0) + 2 \cdot (y - 1) - 4(z - 2) = 0$

2. (20 points) Is the triangle with vertices

$$P = (0,1,2), \quad Q = (2,-1,2), \quad R = (-1,0,1)$$

right-angled?



Compute $\overrightarrow{QP} \cdot \overrightarrow{QR}$, $\overrightarrow{PQ} \cdot \overrightarrow{PR}$, $\overrightarrow{RP} \cdot \overrightarrow{RQ}$

$\overrightarrow{PQ} = \langle 2, -2, 0 \rangle = -\overrightarrow{QP}$

$\overrightarrow{PR} = \langle -1, -1, -1 \rangle = -\overrightarrow{RP}$

$\overrightarrow{QR} = \langle -3, 1, 1 \rangle = -\overrightarrow{RQ}$

$\overrightarrow{QP} \cdot \overrightarrow{QR} = \langle -2, 2, 0 \rangle \cdot \langle -3, 1, 1 \rangle = 6 + 2 = 8$

$\overrightarrow{PQ} \cdot \overrightarrow{PR} = \langle 2, -2, 0 \rangle \cdot \langle -1, -1, -1 \rangle = -2 + 2 = 0$

The triangle PQR is right-angled since vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are orthogonal.

3. (20 points) Compute the volume of the parallelepiped determined by the vectors

$$\mathbf{u} = \langle 1, -1, 2 \rangle, \quad \mathbf{v} = \langle 2, 0, 3 \rangle, \quad \mathbf{w} = \langle 3, -1, 5 \rangle.$$

$$\text{Volume} = \vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 0 & 3 \\ 3 & -1 & 5 \end{vmatrix} = 0$$

4. (20 points) Consider the planes

$$-x - 3y + 2z + 4 = 0$$

and

$$2x + 6y - 2z + 4 = 0.$$

Determine whether these planes are equal, parallel but not equal or intersecting. If they intersect, find the line of intersection between them.

First, find the normal vectors to each plane: $\vec{n_1} = \langle -1, -3, 2 \rangle$

$\vec{n_2} = \langle 2, 6, -2 \rangle$

$\vec{n_1}$ and $\vec{n_2}$ are not parallel, therefore, the two planes intersect.

Let L be the line formed by the intersection. Then the points of L satisfy the equation

$$\begin{cases} -x - 3y + 2z + 4 = 0 & (1) \quad \times 2 \\ 2x + 6y - 2z + 4 = 0 & (2) \end{cases} \quad +$$

$\hookrightarrow$ $4z - 2z + 8 + 4 = 0$, $z = -6$

Now plug $z = -6$ into (1):

$-x - 3y - 12 + 4 = 0$

$x = -3y - 8$

Take $y = t$, then

$x = -3t - 8$

$z = -6$

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

Equation of L: $\begin{cases} x = -3t - 8 \\ y = t \\ z = -6 \end{cases}$