Name (last, first): ____________________

Student ID: ____________________

☐ **Write your name and PID on the top of EVERY PAGE.**

☐ **Write the solutions to each problem on separate pages. CLEARLY INDICATE on the top of each page the number of the corresponding problem. Different parts of the same problem can be written on the same page (for example, part (a) and part (b)).**

☐ **The exam consists of 4 questions. Your answers must be carefully justified to receive credit.**

☐ **This exam will be scanned. Make sure you write ALL SOLUTIONS on the paper provided. DO NOT REMOVE ANY OF THE PAGES.**

☐ **No calculators, phones, or other electronic devices are allowed.**

☐ **Remember this exam is graded by a human being. Write your solutions NEATLY AND COHERENTLY, or they risk not receiving full credit.**

☐ **You are allowed to use one 8.5 by 11 inch sheet of paper with hand-written notes (on both sides); no other notes (or books) are allowed.**

1. (20 points) Let $f(x,y) = (x+y^2)e^{x^2y^2}$.

(a) Find the gradient of the function f at point $(0,0)$. Find the directional derivative of the function f at the point $(0,0)$ in the direction $\vec{v} = \langle 3,4\rangle$.

Compute the partial derivatives at $(0,0)$

$$f_x = e^{x^2y^2} + (x+y^2)e^{x^2y^2}\,2xy^2 = \left(1 + (x+y^2)\,2xy\right)e^{x^2y^2}$$

$$f_x(0,0) = 1\cdot e^0 = 1$$

$$f_y = 2ye^{x^2y^2} + (x+y^2)e^{x^2y^2}\,2yx^2 = 2y\left(1+(x+y^2)x^2\right)e^{x^2y^2}$$

$f_y(0,0) = 0$ Therefore, $\nabla f(0,0) = \langle 1,0\rangle$

In order to compute the directional derivative, we find the unit vector in the direction $\vec{v} = \langle 3,4\rangle$:

$$\|\vec{v}\| = \sqrt{3^2+4^2} = \sqrt{9+16} = \sqrt{25} = 5, \quad \vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \left\langle \frac{3}{5}, \frac{4}{5}\right\rangle$$

Then $D_{\vec{u}}f(0,0) = \nabla f(0,0)\cdot\vec{u} = \langle 1,0\rangle\cdot\left\langle\frac{3}{5},\frac{4}{5}\right\rangle = \frac{3}{5}$

(b) Find the unit vector in the direction of the maximal rate of increase for the function f at the point $(0,0)$. What is the value of the directional derivative in this direction?

The direction of the maximal rate of increase is given by the gradient, $\nabla f(0,0) = \langle 1,0\rangle$.

$\|\nabla f(0,0)\| = \|\langle 1,0\rangle\| = 1$, so $\langle 1,0\rangle$ is the unit vector in the direction of the maximal rate of increase.

$$D_{\nabla f(0,0)}(0,0) = \nabla f(0,0)\cdot\nabla f(0,0) = 1$$

2. (20 points) Use the chain rule to find the partial derivative $\dfrac{\partial z}{\partial v}$ for

$$z = \left(x + \frac{y}{x}\right)^2,$$

where

$$x = u + v, \qquad y = u - v.$$

You may leave your answer as a product of terms, but your answer should not have any derivative operations remaining to be performed. Your final answer should only be a function of u and v.

$$\frac{\partial x}{\partial v} = 1, \quad \frac{\partial y}{\partial v} = -1$$

$$\frac{\partial z}{\partial x} = 2\left(x + \frac{y}{x}\right)\left(1 - \frac{y}{x^2}\right)$$

$$\frac{\partial z}{\partial y} = 2\left(x + \frac{y}{x}\right) \cdot \frac{1}{x}$$

Using the chain rule we have

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$= 2\left(x + \frac{y}{x}\right)\left(1 - \frac{y}{x^2}\right) - 2\left(x + \frac{y}{x}\right) \cdot \frac{1}{x}$$

$$= 2\left(x + \frac{y}{x}\right)\left(1 - \frac{y}{x^2} - \frac{2}{x}\right) = 2\left(u + v + \frac{u-v}{u+v}\right)\left(1 - \frac{u-v}{(u+v)^2} - \frac{2}{u+v}\right)$$

3. (20 points) Find the tangent plane to the function $f(x,y) = \sqrt{xy^2 + \ln(x) + 1}$ at the point $(1,0)$. [Hint. Recall that $\ln(1) = 0$.]

The equation of the tangent plane to f at (x_0, y_0) is given by $z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

Compute each of the unknown coefficients:

$$f(1,0) = \sqrt{1\cdot 0^2 + \ln(1) + 1} = \sqrt{1} = 1$$

$$f_x = \frac{y^2 + \frac{1}{x}}{2\sqrt{xy^2 + \ln x + 1}}, \quad f_x(1,0) = \frac{0^2 + \frac{1}{1}}{2\sqrt{1\cdot 0^2 + \ln(1) + 1}} = \frac{1}{2}$$

$$f_y = \frac{2xy}{2\sqrt{xy^2 + \ln x + 1}}, \quad f_y(1,0) = 0$$

Therefore, the equation of the tangent plane to f at $(1,0)$ is $z = 1 + \frac{1}{2}(x-1)$

4. (20 points) Suppose that $\mathbf{r}(t) = \langle e^{\cos(t)}, \sin(1 - e^{-t}), t\rangle$. Find the unit tangent vector to this curve at time $t = 0$.

Compute the tangent vector at $t=0$

$$\vec{r}\,'(t) = \langle e^{\cos t}(-\sin t), \cos(1-e^{-t})e^{-t}, 1\rangle$$

$$\vec{r}\,'(0) = \langle e^{\cos 0}(-\sin 0), \cos(1-e^{0})e^{0}, 1\rangle$$

$$= \langle 0, 1, 1\rangle$$

$$\|\vec{r}\,'(0)\| = \|\langle 0, 1, 1\rangle\| = \sqrt{0^2+1^2+1^2} = \sqrt{2}$$

The unit tangent vector is

$$\frac{\vec{r}\,'(0)}{\|\vec{r}\,'(0)\|} = \langle 0, \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\rangle.$$